3. THE DENSITY MATRIX DESCRIPTION OF
A TWO-DIMENSIONAL HETERONUCLEAR
CORRELATION SEQUENCE (2DHETCOR)

The purpose of 2DHETCOR is to reveal the pairwise correlation of different nuclear species (e.g., C-H or C-F) in a molecule. This is based on the scalar coupling interaction between the two spins.

3.1 Calculation Steps

Figure I.2 reveals that the density matrix treatment of a pulse sequence must include the following calculation steps:

- thermal equilibrium populations (off diagonal elements are zero)
- effects of rf pulses (rotation operators)
- evolution between pulses
- evolution during acquisition
- determination of observable magnetization.

Applying the sequence to an AX system (nucleus A is a ^{13}C, nucleus X is a proton) we will describe in detail each of these steps.
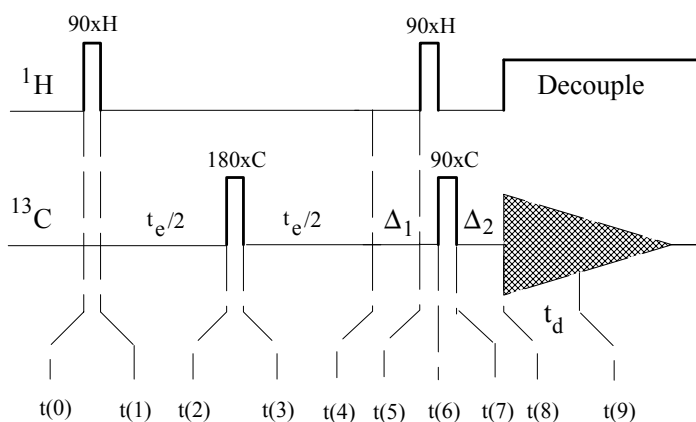

Figure I.2. The two-dimensional heteronuclear correlation sequence:
$90xH - t_e/2 - 180xC - t_e/2 - \Delta_1 - 90xH - 90xC - \Delta_2 - AT$.

3.2 Equilibrium Populations

At thermal equilibrium the four energy levels shown in Figure I.1 are populated according to the Boltzmann distribution law:

$$\frac{P_i}{P_j} = \frac{\exp(-E_i / kT)}{\exp(-E_j / kT)} = \exp\frac{E_j - E_i}{kT} \qquad (I.1)$$

Taking the least populated level as reference we have:

$$P_2 / P_1 = \exp[(E_1 - E_2)/kT] = \exp[h(v_A + J/2)/kT] \quad (I.2)$$

Since transition frequencies (10^8 Hz) are more than six orders of magnitude larger than coupling constants (tens or hundreds of Hz), we may neglect the latter (only when we calculate relative populations; of course, they will not be neglected when calculating transition frequencies). Furthermore, the ratios hv_A/kT and hv_X/kT are   much smaller than 1. For instance, in a 4.7 Tesla magnet the ^{13}C Larmor frequency is $v_A = 50$ x 10^6 Hz and

$$p = \frac{hv_A}{kT} = \frac{6.6 \cdot 10^{-34} \, \text{Js} \cdot 50 \cdot 10^6 \, \text{s}^{-1}}{1.4 \cdot 10^{-23} \, (\text{J/K}) \cdot 300\text{K}} = 0.785 \cdot 10^{-5}$$

This justifies a first order series expansion [see (A11)]:

$$P_2 / P_1 = \exp(hv_A / kT) \cong 1 + (hv_A / kT) = 1 + p \qquad (I.3)$$

$$P_3 / P_1 = \exp(hv_X / kT) \cong 1 + (hv_X / kT) = 1 + q \qquad (I.4)$$

$$P_4 / P_1 \cong 1 + [h(v_A + v_X)/kT] = 1 + p + q \qquad (I.5)$$

In the particular case of the carbon-proton system the Larmor frequencies are in the ratio 1:4 (i.e., $q = 4p$).

We now normalize the sum of populations:

$$P_1 = P_1$$
$$P_2 = (1+p)P_1$$
$$P_3 = (1+4p)P_1$$
$$P_4 = (1+5p)P_1$$

$$\overline{}$$

$$1 = P_1(4+10p) = P_1 S \tag{I.6}$$

Hence,

$$P_1 = 1/S$$
$$P_2 = (1+p)/S$$
$$P_3 = (1+4p)/S$$
$$P_4 = (1+5p)/S$$

where

$$S = 4+10p$$

Given the small value of p we can work with the approximation $S \cong 4$. Then the density matrix at equilibrium is:

$$D(0) = \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & P_2 & 0 & 0 \\ 0 & 0 & P_3 & 0 \\ 0 & 0 & 0 & P_4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+p & 0 & 0 \\ 0 & 0 & 1+4p & 0 \\ 0 & 0 & 0 & 1+5p \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{p}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

It is seen that the first term of the sum above is very large compared to the second term. However, the first term is not important since it contains the unit matrix [see (A20)-(A21)] and is not affected by any evolution operator (see Appendix B). Though much smaller, it is the second term which counts because it contains the population differences (*Vive la difference!*). From now on we will work with this

term only, ignoring the constant factor $p/4$ and taking the license to continue to call it $D(0)$:

$$D(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad (I.7)$$

Equilibrium density matrices for systems other than C-H can be built in exactly the same way.

3.3 The First Pulse

At time $t(0)$ a 90º proton pulse is applied along the x-axis. We now want to calculate $D(1)$, the density matrix after the pulse. The standard formula for this operation,

$$D(1) = R^{-1} D(0) R, \qquad (I.8)$$

is explained in Appendix B.   The rotation operator, $R,$ for this particular case is [see (C18)]:

$$R_{90xH} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{bmatrix} \qquad (I.9)$$

where $i = \sqrt{-1}$ is the imaginary unit.

Its inverse (reciprocal), R^{-1}, is readily calculated by transposition and conjugation [see (A22)-(A23)]:

$$R_{90xH}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{bmatrix} \qquad (I.10)$$

First we multiply $D(0)$ by R. Since the matrix multiplication is not commutative (see Appendix A for matrix multiplication rules), it is necessary to specify that we *postmultiply $D(0)$ by R*:

$$D(0)R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 4i & 0 & 4 & 0 \\ 0 & 5i & 0 & 5 \end{bmatrix} \tag{I.11}$$

Then we *premultiply* the result by R^{-1}:

$$D(1) = R^{-1}[D(0)R] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 4i & 0 & 4 & 0 \\ 0 & 5i & 0 & 5 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 4 & 0 & -4i & 0 \\ 0 & 6 & 0 & -4i \\ 4i & 0 & 4 & 0 \\ 0 & 4i & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2i & 0 \\ 0 & 3 & 0 & -2i \\ 2i & 0 & 2 & 0 \\ 0 & 2i & 0 & 3 \end{bmatrix} \tag{I.12}$$

It is good to check this result by making sure that the matrix $D(1)$ is Hermitian, i.e., every matrix element below the main diagonal is the complex conjugate of its corresponding element above the diagonal [see (A24)] (neither the rotation operators, nor the partial results need be Hermitian). *Comparing $D(1)$ to $D(0)$ we see that the 90º proton pulse created proton single-quantum coherences, did not touch the carbon, and redistributed the populations.*

3.4 Evolution from t (1) to t (2)

The standard formula[1] describing the time evolution of the density matrix elements in the absence of a pulse is:

$$d_{mn}(t) = d_{mn}(0)\exp(-i\omega_{mn}t) \qquad (I.13)$$

d_{mn} is the matrix element (row m, column n) and $\omega_{mn}=(E_m - E_n)/\hbar$ is the angular frequency of the transition $m \rightarrow n$.

We observe that during evolution the diagonal elements are invariant since $\exp[i(E_m - E_m)/\hbar] = 1$. The off diagonal elements experience a periodic evolution. Note that $d_{mn}(0)$ is the starting point of the evolution immediately after a given pulse. In the present case, the elements $d_{mn}(0)$ are those of $D(1)$.

We now want to calculate $D(2)$ at the time $t(2)$ shown in Figure I.2. We have to consider the evolution of elements d_{13} and $d_{24.}$ In a frame rotating with the proton transmitter frequency ω_{trH}, after an evolution time $t_e/2$, their values are:

$$d_{13} = -2i\exp(-i\Omega_{13}t_e/2) = B \qquad (I.14)$$
$$d_{24} = -2i\exp(-i\Omega_{24}t_e/2) = C \qquad (I.15)$$

where $\Omega_{13} = \omega_{13} - \omega_{trH}$ and $\Omega_{24} = \omega_{24} - \omega_{trH}$.

Hence

$$D(2) = \begin{bmatrix} 2 & 0 & B & 0 \\ 0 & 3 & 0 & C \\ B* & 0 & 2 & 0 \\ 0 & C* & 0 & 3 \end{bmatrix} \qquad (I.16)$$

$B*$ and $C*$ are the complex conjugates of B and C (see Appendix A).

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[1]In our treatment, relaxation during the pulse sequence is ignored. This contributes to a significant simplification of the calculations without affecting the main features of the resulting 2D spectrum.

3.5 The Second Pulse

The rotation operators for this pulse are [see(C17)]:

$$R_{180xC} = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} \quad \text{(I.17)}; \quad R_{180xC}^{-1} = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix} \quad \text{(I.18)}$$

Postmultiplying $D(2)$ by R gives:

$$D(2)R_{180xC} = \begin{bmatrix} 0 & 2i & 0 & iB \\ 3i & 0 & iC & 0 \\ 0 & i & iB* & 2i \\ iC* & 0 & 3i & 0 \end{bmatrix} \quad \text{(I.19)}$$

Premultiplying (I.19) by R^{-1} gives:

$$D(3) = \begin{bmatrix} 3 & 0 & C & 0 \\ 0 & 2 & 0 & B \\ C* & 0 & 3 & 0 \\ 0 & B* & 0 & 2 \end{bmatrix} \quad \text{(I.20)}$$

Comparing $D(3)$ with $D(2)$ we note that the $180°$ pulse on carbon has caused a population inversion (interchange of d_{11} and d_{22}). It has also interchanged the coherences B and C (d_{13} and d_{24}). This means that B, after having evolved with the frequency ω_{13} during the first half of the evolution time [see (I.14)], will now evolve with the frequency ω_{24}, while C switches form ω_{24} to ω_{13}.

3.6 Evolution from t (3) to t (4)

According to (I.13) the elements d_{13} and d_{42} become:

$$d_{13} = C\exp(-i\Omega_{13}t_e/2) \tag{I.21}$$
$$d_{24} = B\exp(-i\Omega_{24}t_e/2) \tag{I.22}$$

From Figure I.1 we see that in the laboratory frame

$$\omega_{13} = 2\pi(\nu_X - J/2) = \omega_H + \pi J \tag{I.23}$$
$$\omega_{24} = 2\pi(\nu_X - J/2) = \omega_H - \pi J \tag{I.24}$$

In the rotating frame (low case) ω becomes (capital) Ω. Taking the expressions of B and C from (I.14) and (I.15), relations (I.21) and (I.22) become

$$d_{13} = -2i\exp[-i(\Omega_H - \pi J)t_e/2]\exp[-i(\Omega_H + \pi J)t_e/2]$$
$$= -2i\exp(-i\Omega_H t_e) \tag{I.25}$$
$$d_{24} = -2i\exp(-i\Omega_H t_e) = d_{13} \tag{I.26}$$

None of the matrix elements of $D(4)$ contains the coupling constant J. The result looks like that of a *decoupled evolution*. The averaged shift Ω_H (center frequency of the doublet) is expressed while the coupling is not. We know that the coupling J was actually present during the evolution, as documented by the intermediate results $D(2)$ and $D(3)$. We call the sequence $t_e/2$ - $180C$ - $t_e/2$ a *refocusing routine*. The protons which were fast (Ω_{13}) during the first $t_e/2$ are slow (Ω_{24}) during the second $t_e/2$ and vice versa (they change label).

3.7 The Role of Δ_1

In order to understand the role of the supplementary evolution Δ_1 we have to carry on the calculations without it, i.e., with $d_{13}=d_{24}$. We find out (see Appendix I) that the useful signal is canceled. To obtain maximum signal, d_{13} and d_{24} must be equal but of opposite signs. This is what the delay Δ_1 enables us to achieve.