Preface

This book grew out of lecture notes from a mini-course I gave at the 2014 Women and Mathematics program at the Institute for Advanced Study. When asked to provide background reading for the participants, I found myself at a bit of a loss; while there are many excellent books which give some treatment of Haar distributed random matrices, there was no one source that gave a broad, and broadly accessible, introduction to the subject in its own right. My goal has been to fill this gap: to give an introduction to the theory of random orthogonal, unitary, and symplectic matrices which approaches the subject from many angles, includes the most important results that anyone looking to learn about the subject should know, and tells a coherent story that allows the beauty of this many-faceted subject to shine through.

The book begins with a very brief introduction to the orthogonal, unitary, and symplectic groups; just enough to get started talking about Haar measure. The second section includes six different constructions of Haar measure on the classical groups; the chapter also contains some further information on the groups, including some basic aspects of their structure as Lie groups, identification of the Lie algebras, an introduction to representation theory, and discussion of the characters.

Chapter two is about the joint distribution of the entries of a Haar-distributed random matrix. The fact that individual entries are approximately Gaussian is classical and goes back to the late 19^{th} century. This chapter includes modern results on the joint distribution of the entries in various senses: total variation approximation of principal submatrices by Gaussian matrices, in-probability approximation of (much larger) submatrices by Gaussian matrices, and a treatment of arbitrary projections of Haar measure via Stein's method.

Chapters three and four deal with the eigenvalues. Chapter three is all about exact formulas: the Weyl integration formulas, the structure of the eigenvalue processes as determinantal point processes with explicit kernels, exact formulas due to Diaconis and Shahshahani for the matrix moments, and an interesting decomposition (due to Eric Rains) of the distribution of eigenvalues of powers of random matrices.

Chapter four deals with asymptotics for the eigenvalues of large matrices: the sine kernel microscopic scaling limit, limit theorems for the empirical spectral measures and linear eigenvalue statistics, large deviations for the empirical spectral measures, and an interesting self-similarity property of the eigenvalue distribution.

Chapters five and six are where this project began: concentration of measure on the classical compact groups, with applications in geometry. Chapter five introduces the concept of concentration of measure, the connection with log-Sobolev inequalities, and derivations of optimal (at least up to constants) log-Sobolev constants. The final section contains concentration inequalities for the empirical spectral measures of random unitary matrices.

Chapter six has some particularly impressive applications of measure concentration on the classical groups to high-dimensional geometry. First, a proof of the celebrated Johnson–Lindenstrauss lemma via concentration of measure on the orthogonal group, with a (very brief) discussion of the role of the lemma in randomized algorithms. The second section is devoted to a proof of Dvoretzky's theorem, again via concentration of measure (this time on the unitary group). The final section gives the proof of a "measure-theoretic" Dvoretzky theorem, showing that subject to some mild constraints, most marginals of high-dimensional probability measures are close to Gaussian.

Finally, chapter seven gives a taste of the intriguing connection between eigenvalues of random unitary matrices and zeros of the Riemann zeta function. There is a section on Montgomery's theorem and conjecture on pair correlations and one on the results of Keating and Snaith on the characteristic polynomial of a random unitary matrix, which led them to exciting new conjectures on the zeta side. Some numerical evidence (and striking pictures) are presented.

I have tried to make the book accessible to a reader with an undergraduate background in mathematics generally, with a bit more in probability (e.g., comfort with measure theory would be good). But because the random matrix theory of the classical compact groups touches on so many diverse areas of mathematics, it has been my assumption in writing this book that most readers will not be familiar with all of the background which comes up. I have done my best to give accessible, bottom-line introductions to the areas I thought were most likely to be unfamiliar, but there are no doubt places where an unfamiliar (or more likely, vaguely familiar, but without enough associations for comfort) phrase will suddenly appear. In these cases, it seems best to take the advice of John von Neumann, who said to a student "... in mathematics you don't understand things. You just get used to them."

One of the greatest pleasures in completing a book is the opportunity to thank the many sources of knowledge, advice, wisdom, and support that made it possible. My thanks firstly to the Institute for Advanced Study and the organizers of the Women and Mathematics program for inviting me to give the lectures that inspired this book. Thanks also to the National Science Foundation for generous support while I wrote it.

Amir Dembo encouraged me to embark on this project and gave me valuable advice about how to do it well.

I am grateful to Pierre Albin and Tyler Lawson for their constant willingness to patiently answer all of my questions about geometry and algebra, and if they didn't already know the answers, to help me wade through unfamiliar literature. Experienced guides make all the difference.

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The excellent and topical photograph on the frontispiece was found (I still don't know how) by Tim Gowers.

As ever, thanks to Sarah Jarosz, this time for *Undercurrent*, which got me most of the way there, and to Yo-Yo Ma for *Six Evolutions*, which carried me to the finish line.

And how to thank my husband and collaborator, Mark Meckes? We have discussed the material in this book for so long and in so many contexts that his viewpoint is inextricably linked with my own. He has lived with the writing of this book, always willing to drop a (probably more important) conversation or task in order to let me hash out a point that suddenly felt terribly urgent. If my writing helps to illuminate the ideas I have tried to describe, it is because I got to talk it out first at the breakfast table.