Random Matrices from the Classical Compact Groups

Elizabeth Meckes

Case Western Reserve University

ILAS 2016, Leuven

The classical compact matrix groups

The classical compact matrix groups

- $\mathbb{O}(n)$: $U \in M_n(\mathbb{R})$, $UU^T = I_n$

The classical compact matrix groups

- $\mathbb{O}(n)$: $U \in M_n(\mathbb{R})$, $UU^T = I_n$
  - $\mathbb{SO}(n)$: $U \in \mathbb{O}(n)$, $\det(U) = 1$

The classical compact matrix groups

- $\mathbb{O}(n)$: $U \in M_n(\mathbb{R})$, $UU^T = I_n$
  - $\mathbb{SO}(n)$: $U \in \mathbb{O}(n)$, $\det(U) = 1$
- $\mathbb{U}(n)$: $U \in M_n(\mathbb{C})$, $UU^* = I_n$

The classical compact matrix groups

- $\mathbb{O}(n)$: $U \in M_n(\mathbb{R})$, $UU^T = I_n$
  - $\mathbb{SO}(n)$: $U \in \mathbb{O}(n)$, $\det(U) = 1$
- $\mathbb{U}(n)$: $U \in M_n(\mathbb{C})$, $UU^* = I_n$
  - $\mathbb{SU}(n)$: $U \in \mathbb{U}(n)$, $\det(U) = 1$

The classical compact matrix groups

- $\mathbb{O}(n)$: $U \in M_n(\mathbb{R})$, $UU^T = I_n$
  - $\mathbb{SO}(n)$: $U \in \mathbb{O}(n)$, $\det(U) = 1$
- $\mathbb{U}(n)$: $U \in M_n(\mathbb{C})$, $UU^* = I_n$
  - $\mathbb{SU}(n)$: $U \in \mathbb{U}(n)$, $\det(U) = 1$
- $\mathbb{Sp}(2n)$: $U \in M_n(\mathbb{H})$, $UU^* = I_n$

The classical compact matrix groups

- $\mathbb{O}(n)$: $U \in M_n(\mathbb{R})$, $UU^T = I_n$
  - $\mathbb{SO}(n)$: $U \in \mathbb{O}(n)$, $\det(U) = 1$
- $\mathbb{U}(n)$: $U \in M_n(\mathbb{C})$, $UU^* = I_n$
  - $\mathbb{SU}(n)$: $U \in \mathbb{U}(n)$, $\det(U) = 1$
- $\mathbb{Sp}(2n)$: $U \in M_n(\mathbb{H})$, $UU^* = I_n$
  - Alternatively: $U \in \mathbb{U}(2n)$, $UJU^* = J$, with $J = \begin{bmatrix} \mathbf{0} & I_n \\ -I_n & \mathbf{0} \end{bmatrix}$

Random matrices

Random matrices

Why?

Random matrices

Why?

- Curiosity: We understand the matrix groups better if we know what a random element is like.

Random matrices

Why?

- Curiosity: We understand the matrix groups better if we know what a random element is like.
- Randomized algorithms: Sometimes any random thing will do the job (but it's still hard to write a deterministic algorithm!)

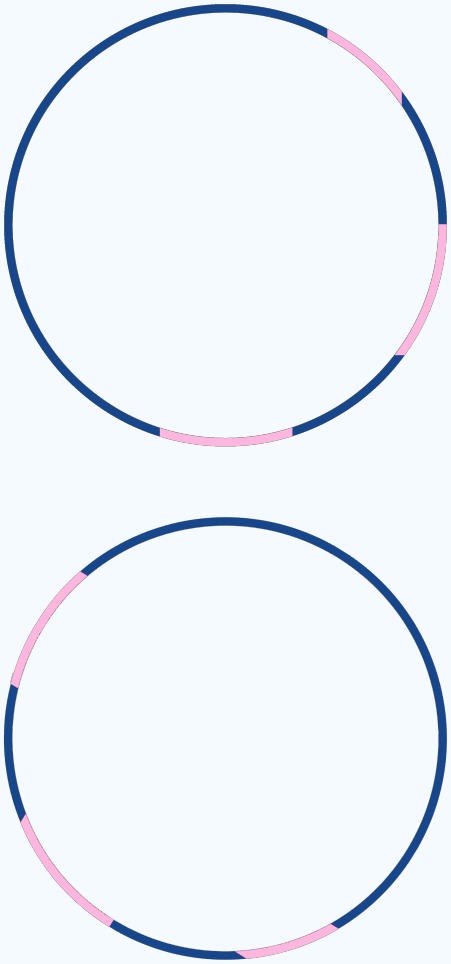
Random matrices

Okay, how?

Random matrices

Okay, how?

Haar measure: On each group, there is a unique translation-invariant probability measure.



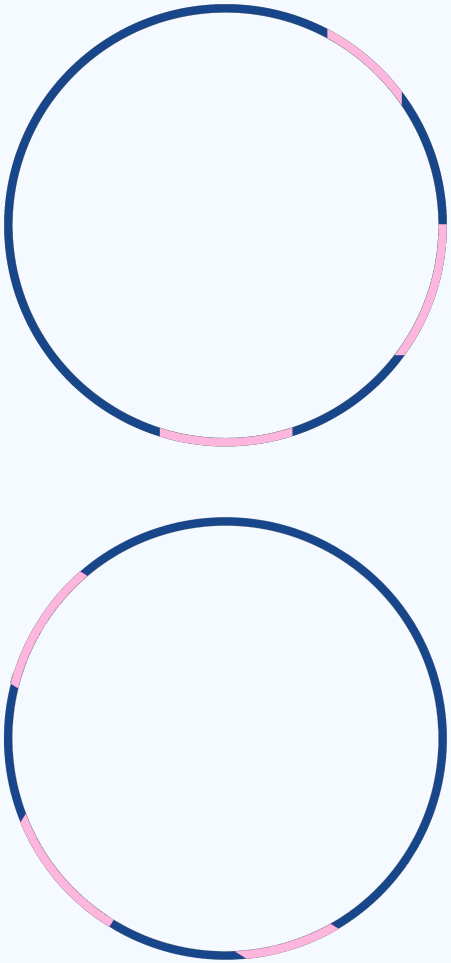
Random matrices

Okay, how?

Haar measure: On each group, there is a unique translation-invariant probability measure.

U is a Haar random matrix on $\mathbb{O}(n)$ iff for $S \subseteq \mathbb{O}(n)$ and any deterministic $A \in \mathbb{O}(n)$:

$$\mathbb{P}[U \in S] = \mathbb{P}[U \in A \cdot S] = \mathbb{P}[U \in S \cdot A]$$



Random matrices

Okay, how?

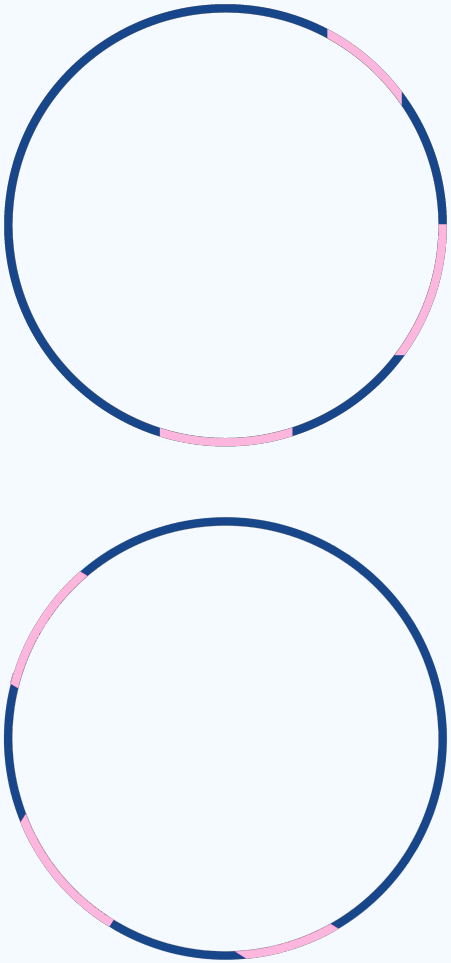
Haar measure: On each group, there is a unique translation-invariant probability measure.

U is a Haar random matrix on $\mathbb{O}(n)$ iff for $S \subseteq \mathbb{O}(n)$ and any deterministic $A \in \mathbb{O}(n)$:

$$\mathbb{P}[U \in S] = \mathbb{P}[U \in A \cdot S] = \mathbb{P}[U \in S \cdot A]$$

$$\Longleftrightarrow$$

$$U \stackrel{d}{=} AU \stackrel{d}{=} UA$$



Random matrices

But how, really?

Random matrices

But how, really?

- Fill an empty $n \times n$ matrix with i.i.d. Gaussians, and perform the Gram-Schmidt process.

Random matrices

But how, really?

- Fill an empty $n \times n$ matrix with i.i.d. Gaussians, and perform the Gram-Schmidt process.
- Fill the first column of a matrix with a vector chosen uniformly from the sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. Then fill the second column with a vector chosen uniformly in the orthogonal complement of the first. And so on.

Eigenvalues

Eigenvalues

Matrices from the classical compact groups have eigenvalues which lie on the unit circle in the complex plane.

Eigenvalues

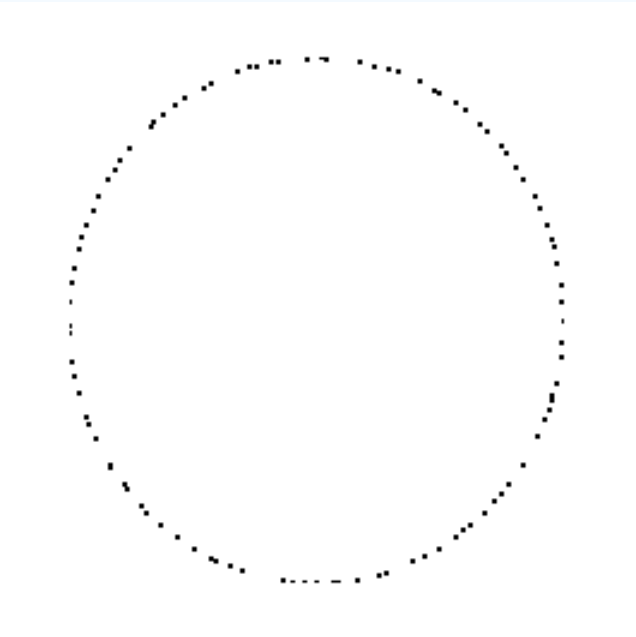
Matrices from the classical compact groups have eigenvalues which lie on the unit circle in the complex plane.

They are distinct with probability one, but there's more:

Eigenvalues

Matrices from the classical compact groups have eigenvalues which lie on the unit circle in the complex plane.

They are distinct with probability one, but there's more:

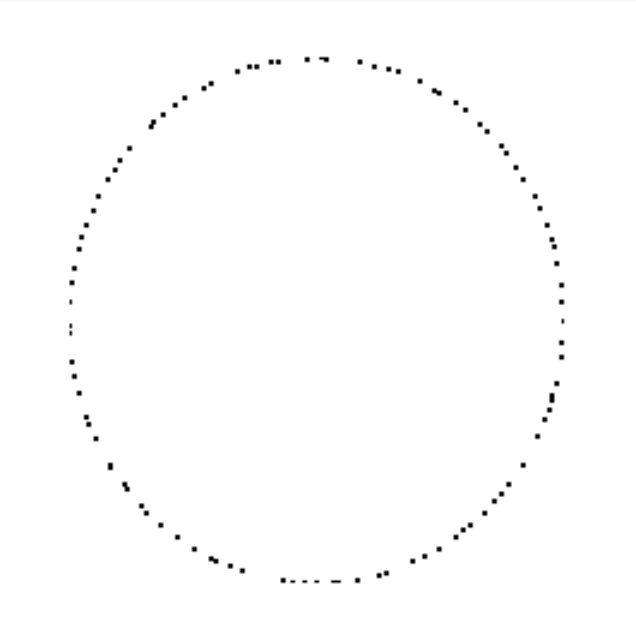


The eigenvalues of a
100×100 random unitary matrix

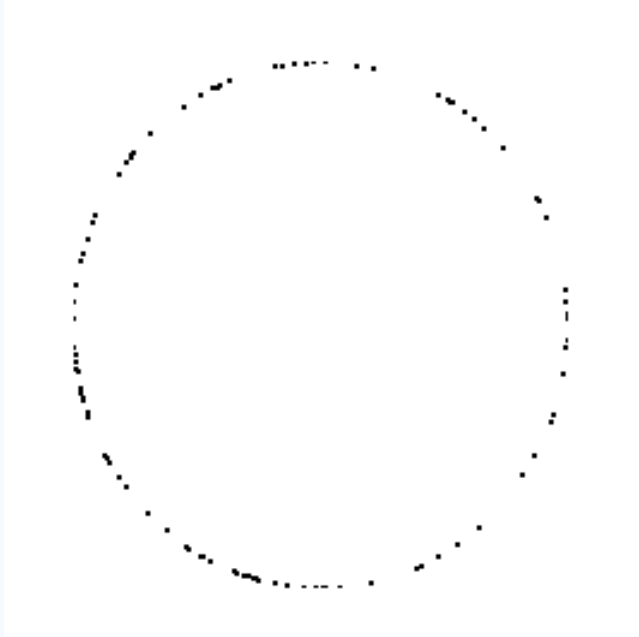
Eigenvalues

Matrices from the classical compact groups have eigenvalues which lie on the unit circle in the complex plane.

They are distinct with probability one, but there's more:



The eigenvalues of a 100×100 random unitary matrix



E. Rains

100 i.i.d. uniform random points

The empirical spectral measure

The empirical spectral measure

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a random matrix U.

The empirical spectral measure

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a random matrix U.

The empirical spectral measure μ_n is the probability measure

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}.$$

The empirical spectral measure

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a random matrix U.

The empirical spectral measure μ_n is the probability measure

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}.$$

The empirical spectral measure is a handy way to encode the set of eigenvalues as one object to study.

Limiting eigenvalue densities via the empirical spectral measure

Limiting eigenvalue densities via the empirical spectral measure

The semi-circle law: The empirical spectral measure of a Wigner random matrix converges weakly almost surely to the semi-circular distribution $\frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}_{[-2,2]}(x)dx$.

Limiting eigenvalue densities via the empirical spectral measure

The semi-circle law: The empirical spectral measure of a Wigner random matrix converges weakly almost surely to the semi-circular distribution $\frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}_{[-2,2]}(x)dx$.

Roughly, if A is an $n \times n$ Wigner random matrix and n is large, then if $(\alpha, \beta) \subseteq [-2, 2]$,

$$\frac{\#\{\text{eigenvalues of } A \text{ in } (\alpha,\beta)\}}{n} \approx \frac{1}{2\pi}\int_{\alpha}^{\beta}\sqrt{4-x^2}dx.$$

Distances between probability measures

Distances between probability measures

Measuring the distance between a measure supported on n points on the circle and the uniform measure on the circle is a way to quantify how regularly the points are spaced.

Distances between probability measures

Measuring the distance between a measure supported on n points on the circle and the uniform measure on the circle is a way to quantify how regularly the points are spaced.

The L_1 Kantorivich distance is one of many metrics on the set of probability measures:

Distances between probability measures

Measuring the distance between a measure supported on n points on the circle and the uniform measure on the circle is a way to quantify how regularly the points are spaced.

The L_1 Kantorivich distance is one of many metrics on the set of probability measures:

For probabilities μ and ν on a space X,

$$W_1(\mu,\nu) = \inf_{\substack{\pi(A\times X)=\mu(A)\\ \pi(X\times A)=\nu(A)}} \int |x-y| d\pi(x,y)$$

Distances between probability measures

Measuring the distance between a measure supported on n points on the circle and the uniform measure on the circle is a way to quantify how regularly the points are spaced.

The L_1 Kantorivich distance is one of many metrics on the set of probability measures:

For probabilities μ and ν on a space X,

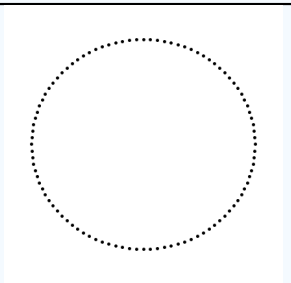
$$W_1(\mu,\nu) = \inf_{\substack{\pi(A\times X)=\mu(A)\\ \pi(X\times A)=\nu(A)}} \int |x-y| d\pi(x,y)$$

$$= \sup_{|f|_L \le 1} \left| \int f(x) d\mu(x) - \int f(x) d\nu(x) \right|.$$

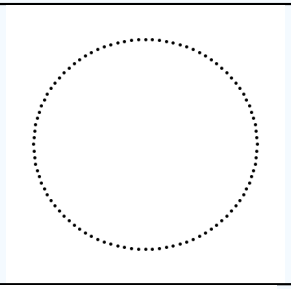
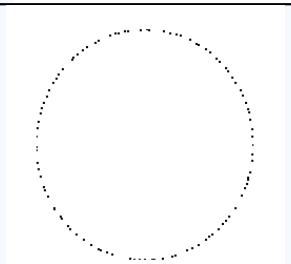
Eigenvalue repulsion quantified

Source of points		Distance to uniform

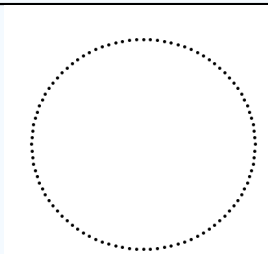
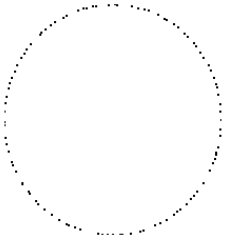
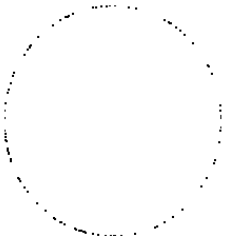
Eigenvalue repulsion quantified

Source of points		Distance to uniform
The picket fence		$\frac{1}{n}$

Eigenvalue repulsion quantified

Source of points		Distance to uniform
The picket fence		$\frac{1}{n}$
Eigenvalues		$\frac{\sqrt{\log(n)}}{n}$

Eigenvalue repulsion quantified

Source of points		Distance to uniform
The picket fence		$\frac{1}{n}$
Eigenvalues		$\frac{\sqrt{\log(n)}}{n}$
i.i.d. uniform		$\frac{1}{\sqrt{n}}$

Distribution of the entries

Distribution of the entries

Heuristically, a Haar random matrix is kind of like a matrix of i.i.d. Gaussians:

Distribution of the entries

Heuristically, a Haar random matrix is kind of like a matrix of i.i.d. Gaussians:

- All the entries have the same individual distributions, and all are roughly Gaussian (mean 0 and variance $\frac{1}{n}$ in $\mathbb{O}(n)$) when n is large.

Distribution of the entries

Heuristically, a Haar random matrix is kind of like a matrix of i.i.d. Gaussians:

- All the entries have the same individual distributions, and all are roughly Gaussian (mean 0 and variance $\frac{1}{n}$ in $\mathbb{O}(n)$) when n is large.
- The entries aren't too dependent.

Theorem (T. Jiang)

Let X be an $n \times n$ matrix of i.i.d. Gaussians, and let U be the result of performing the Gram-Schmidt process on X, so that U is a Haar random orthogonal matrix.

Theorem (T. Jiang)

Let X be an $n \times n$ matrix of i.i.d. Gaussians, and let U be the result of performing the Gram-Schmidt process on X, so that U is a Haar random orthogonal matrix. If

$$\epsilon_n(m) = \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} \left| \sqrt{n} u_{ij} - x_{ij} \right|,$$

Theorem (T. Jiang)

Let X be an $n \times n$ matrix of i.i.d. Gaussians, and let U be the result of performing the Gram-Schmidt process on X, so that U is a Haar random orthogonal matrix. If

$$\epsilon_n(m) = \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} \left| \sqrt{n} u_{ij} - x_{ij} \right|,$$

then $\epsilon_n(m_n) \xrightarrow[n \to \infty]{\mathbb{P}} 0$ *if and only if* $m_n = o\left(\frac{n}{\log(n)}\right)$.

Theorem (T. Jiang)

Let X be an $n \times n$ matrix of i.i.d. Gaussians, and let U be the result of performing the Gram-Schmidt process on X, so that U is a Haar random orthogonal matrix. If

$$\epsilon_n(m) = \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} \left| \sqrt{n} u_{ij} - x_{ij} \right|,$$

then $\epsilon_n(m_n) \xrightarrow[n \to \infty]{\mathbb{P}} 0$ *if and only if* $m_n = o\left(\frac{n}{\log(n)}\right)$.

Bottom line: in this rather weak sense, a random orthogonal matrix is like a matrix of i.i.d. Gaussians, as long as you only consider the first $o\left(\frac{n}{\log(n)}\right)$ columns.

Theorem (Chatterjee–M.)

Let $U \in \mathbb{O}(n)$ be a random orthogonal matrix, let $A_1, \ldots, A_k \in \mathbb{O}(n)$ be orthonormal (w.r.t. $\langle A, B \rangle = \mathrm{Tr}(AB^T)$), and let

$$X = \big(\mathrm{Tr}(A_1 U), \ldots, \mathrm{Tr}(A_k U) \big).$$

Theorem (Chatterjee–M.)

Let $U \in \mathbb{O}(n)$ be a random orthogonal matrix, let $A_1, \ldots, A_k \in \mathbb{O}(n)$ be orthonormal (w.r.t. $\langle A, B \rangle = \mathrm{Tr}(AB^T)$), and let

$$X = \big(\mathrm{Tr}(A_1 U), \ldots, \mathrm{Tr}(A_k U) \big).$$

Let $Z = (Z_1, \ldots, Z_k)$ a vector of i.i.d. standard Gaussians. Then

$$W_1(X, Z) \leq \frac{\sqrt{2}k}{n-1}.$$

Theorem (Chatterjee–M.)

Let $U \in \mathbb{O}(n)$ be a random orthogonal matrix, let $A_1, \ldots, A_k \in \mathbb{O}(n)$ be orthonormal (w.r.t. $\langle A, B \rangle = \mathrm{Tr}(AB^T)$), and let

$$X = \big(\mathrm{Tr}(A_1 U), \ldots, \mathrm{Tr}(A_k U) \big).$$

Let $Z = (Z_1, \ldots, Z_k)$ a vector of i.i.d. standard Gaussians. Then

$$W_1(X, Z) \leq \frac{\sqrt{2}k}{n-1}.$$

Bottom line: In this stronger sense, a random matrix is like a matrix of i.i.d. Gaussians at rank $o(n)$.

Concentration of measure

Concentration of measure

The idea: If you're lucky, "typical" is the same as "average" (averages are easier!).

Concentration of measure

The idea: If you're lucky, "typical" is the same as "average" (averages are easier!).

Theorem
Let G_n be one of $\mathbb{SO}(n), \mathbb{SO}^-(n), \mathbb{SU}(n), \mathbb{U}(n), \mathbb{Sp}(2n)$, and let $F : G_n \to \mathbb{R}$ be 1-Lipschitz.

Concentration of measure

The idea: If you're lucky, "typical" is the same as "average" (averages are easier!).

Theorem

Let G_n be one of $\mathbb{SO}(n), \mathbb{SO}^-(n), \mathbb{SU}(n), \mathbb{U}(n), \mathbb{Sp}(2n)$, and let $F : G_n \to \mathbb{R}$ be 1-Lipschitz. If U is a Haar random matrix in G_n, then

$$\mathbb{P}\left[\left|F(U) - \mathbb{E}F(U)\right| > t\right] \leq Ce^{-cnt^2}.$$

Concentration of the empirical spectral measure

Concentration of the empirical spectral measure

Consider the function $F : G_n \to \mathbb{R}$ defined by

$$F(U) = W_1(\mu_U, \nu).$$

Concentration of the empirical spectral measure

Consider the function $F : G_n \to \mathbb{R}$ defined by

$$F(U) = W_1(\mu_U, \nu).$$

It follows from the Hoffman-Wieland inequality that F is $\frac{1}{\sqrt{n}}$-Lipschitz:

Concentration of the empirical spectral measure

Consider the function $F : G_n \to \mathbb{R}$ defined by

$$F(U) = W_1(\mu_U, \nu).$$

It follows from the Hoffman-Wieland inequality that F is $\frac{1}{\sqrt{n}}$-Lipschitz:

$$|W_1(\mu_U, \nu) - W_1(\mu_V, \nu)| \leq W_1(\mu_U, \mu_V)$$

Concentration of the empirical spectral measure

Consider the function $F : G_n \to \mathbb{R}$ defined by

$$F(U) = W_1(\mu_U, \nu).$$

It follows from the Hoffman-Wieland inequality that F is $\frac{1}{\sqrt{n}}$-Lipschitz:

$$\begin{aligned} |W_1(\mu_U, \nu) - W_1(\mu_V, \nu)| &\leq W_1(\mu_U, \mu_V) \\ &\leq \inf_{\sigma \in S_n} \frac{1}{n} \sum_{j=1}^{n} |\lambda_j(U) - \lambda_{\sigma(j)}(V)| \end{aligned}$$

Concentration of the empirical spectral measure

Consider the function $F : G_n \to \mathbb{R}$ defined by

$$F(U) = W_1(\mu_U, \nu).$$

It follows from the Hoffman-Wieland inequality that F is $\frac{1}{\sqrt{n}}$-Lipschitz:

$$\begin{aligned} |W_1(\mu_U, \nu) - W_1(\mu_V, \nu)| &\leq W_1(\mu_U, \mu_V) \\ &\leq \inf_{\sigma \in S_n} \frac{1}{n} \sum_{j=1}^{n} |\lambda_j(U) - \lambda_{\sigma(j)}(V)| \\ &\leq \frac{1}{\sqrt{n}} \|U - V\|_{H.S.}. \end{aligned}$$

Concentration of the empirical spectral measure

By concentration of measure, this means

$$\mathbb{P}\Big[\big|W_1(\mu_n,\nu) - \mathbb{E}W_1(\mu_n,\nu)\big| > t\Big] \leq Ce^{-cn^2t^2}.$$

Concentration of the empirical spectral measure

By concentration of measure, this means

$$\mathbb{P}\Big[\big|W_1(\mu_n,\nu) - \mathbb{E}W_1(\mu_n,\nu)\big| > t\Big] \leq Ce^{-cn^2t^2}.$$

$\Longrightarrow$ $W_1(\mu_n,\nu)$ is typically within about $\frac{1}{n}$ of its mean.

The Johnson–Lindenstrauss Lemma

The Johnson–Lindenstrauss Lemma

If you have n high-dimensional data points and project them onto a random subspace of dimension $\sim \log(n)$, the pairwise distances between the points is approximately preserved.

The Johnson–Lindenstrauss Lemma

If you have n high-dimensional data points and project them onto a random subspace of dimension $\sim \log(n)$, the pairwise distances between the points is approximately preserved.

Practical conclusion: If your problem is about the metric structure of the data (finding the closest pair, most separated pair, minimum spanning tree of a graph,etc.), there is no need to work in the high-dimensional space that the data naturally live in.

The Johnson–Lindenstrauss Lemma

Lemma (J–L)

Let $\{x_j\}_{j=1}^{n} \subseteq \mathbb{R}^d$, let U be a random $d \times d$ orthogonal and let P be the $k \times d$ matrix which is the first k rows of U; that is,

P is a projection of $\mathbb{R}^d$ onto a random k-dimensional subspace.

The Johnson–Lindenstrauss Lemma

Lemma (J–L)

Let $\{x_j\}_{j=1}^n \subseteq \mathbb{R}^d$, let U be a random $d \times d$ orthogonal and let P be the $k \times d$ matrix which is the first k rows of U; that is,

P is a projection of $\mathbb{R}^d$ onto a random k-dimensional subspace.

If $k = \frac{a \log(n)}{\epsilon^2}$, then with probability at least $1 - \frac{C}{n^{\frac{ac}{9}-2}}$ (with C, c coming from the concentration inequality),

$$(1-\epsilon)\|x_i - x_j\|^2 \leq \left(\frac{d}{k}\right)\|Px_i - Px_j\|^2 \leq (1+\epsilon)\|x_i - x_j\|^2$$

for all $i, j \in \{1, \ldots, n\}$.

Application: Finding the closest point

Application: Finding the closest point

Consider the following problem: You are given a reference set $\mathcal{X}$ of n points in $\mathbb{R}^d$. Now given a query point $q \in \mathbb{R}^d$, find the closest point in $\mathcal{X}$ to q.

P. Indyk

dimension = number of pixels

Application: Finding the closest point

Consider the following problem: You are given a reference set $\mathcal{X}$ of n points in $\mathbb{R}^d$. Now given a query point $q \in \mathbb{R}^d$, find the closest point in $\mathcal{X}$ to q.

P. Indyk

dimension = number of pixels

The naïve approach – calculate each distance and keep track of the best so far – runs in $O(nd)$ steps.

Application: Finding the closest point

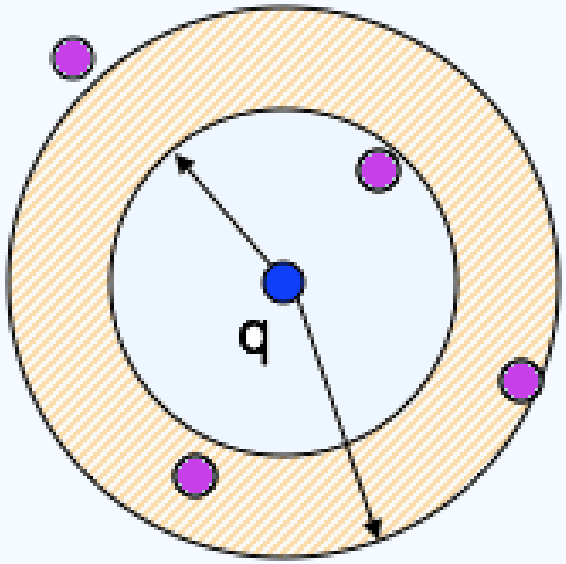
Relaxing the problem:

Application: Finding the closest point

Relaxing the problem:

If you project onto a random subspace of dimension about $\log(n)$, distances are approximately preserved.

This means that while the algorithm might not return the absolute closest point, the point that it returns will be almost as close to q as the true closest point is.



P. Indyk

More carefully, suppose that P is one of the good random projections so that

$$(1-\epsilon)\|q-x_i\|^2 \leq \left(\frac{d}{k}\right)\|Pq-Px_i\|^2 \leq (1+\epsilon)\|q-x_i\|^2$$

for each i.

More carefully, suppose that P is one of the good random projections so that

$$(1-\epsilon)\|q-x_i\|^2 \leq \left(\frac{d}{k}\right)\|Pq-Px_i\|^2 \leq (1+\epsilon)\|q-x_i\|^2$$

for each i.

If Px_i is the closest point to Pq (and so our randomized algorithm returns x_i), but the true closest point to q is x_j, then

$$\|q-x_i\| \leq \sqrt{\frac{1+\epsilon}{1-\epsilon}}\|q-x_j\|;$$

that is, the wrong answer isn't *that* wrong.

More carefully, suppose that P is one of the good random projections so that

$$(1-\epsilon)\|q-x_i\|^2 \leq \left(\frac{d}{k}\right)\|Pq-Px_i\|^2 \leq (1+\epsilon)\|q-x_i\|^2$$

for each i.

If Px_i is the closest point to Pq (and so our randomized algorithm returns x_i), but the true closest point to q is x_j, then

$$\|q-x_i\| \leq \sqrt{\frac{1+\epsilon}{1-\epsilon}}\|q-x_j\|;$$

that is, the wrong answer isn't *that* wrong.

And after projecting, the naïve approach runs in $O(n\log(n))$ steps, instead of $O(n^2)$.

Other neat stuff: powers of random matrices

Other neat stuff: powers of random matrices

Theorem (Rains 1997)

Let $U \in \mathbb{U}(n)$ be a random unitary matrix, and let $m \geq n$. Then the eigenvalues of U^m are distributed exactly *as n i.i.d. uniform points on $\mathbb{S}^1$.*

Other neat stuff: powers of random matrices

Theorem (Rains 1997)

Let $U \in \mathbb{U}(n)$ be a random unitary matrix, and let $m \geq n$. Then the eigenvalues of U^m are distributed exactly *as n i.i.d. uniform points on $\mathbb{S}^1$.*

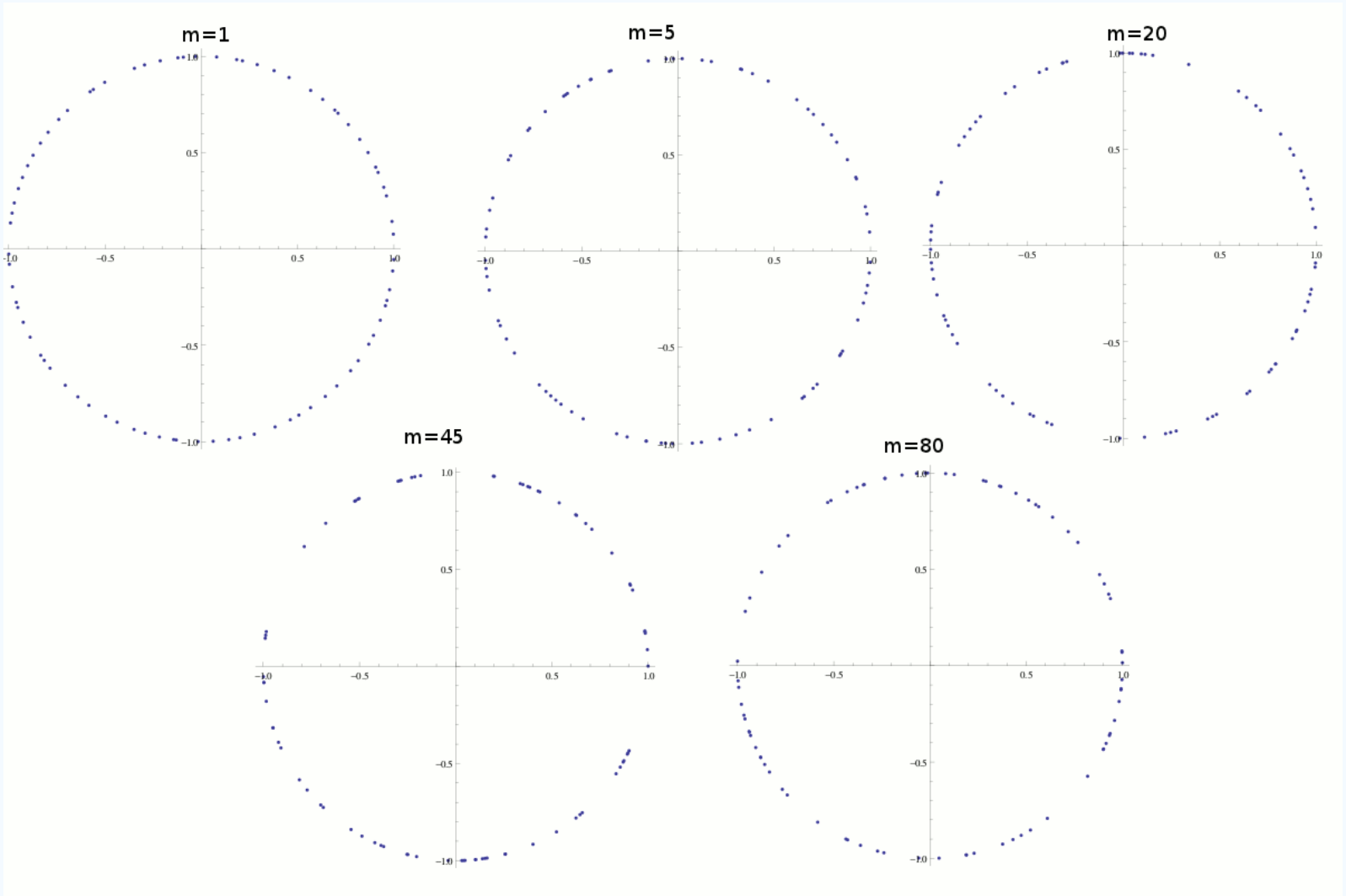
Theorem (Rains 2003)

Let $m \leq N$ be fixed. Then

$$[\mathbb{U}(N)]^m \overset{e.v.d.}{=} \bigoplus_{0 \leq j < m} \mathbb{U}\left(\left\lceil \frac{N-j}{m} \right\rceil\right),$$

where $\overset{e.v.d.}{=}$ denotes equality of eigenvalue distributions.

Other neat stuff: powers of random matrices



The eigenvalues of U^m for $m = 1, 5, 20, 45, 80$, for U a realization of a random 80×80 unitary matrix.

Other neat stuff: self-similarity

Other neat stuff: self-similarity

- Let $\mathcal{N}_\theta$ be the number of eigenvalue angles of an $n \times n$ random unitary matrix in $[-\theta, \theta) \subseteq [-\pi, \pi)$.

Other neat stuff: self-similarity

- Let $\mathcal{N}_\theta$ be the number of eigenvalue angles of an $n \times n$ random unitary matrix in $[-\theta, \theta) \subseteq [-\pi, \pi)$.
- Take a random $nm \times nm$ unitary matrix, and zoom in on $\left[-\frac{\pi}{m}, \frac{\pi}{m}\right)$: let $\mathcal{N}_\theta^{(m)}$ be the number of eigenvalue angles in $\left[-\frac{\theta}{m}, \frac{\theta}{m}\right)$.

Other neat stuff: self-similarity

- Let $\mathcal{N}_\theta$ be the number of eigenvalue angles of an $n \times n$ random unitary matrix in $[-\theta, \theta) \subseteq [-\pi, \pi)$.
- Take a random $nm \times nm$ unitary matrix, and zoom in on $\left[-\frac{\pi}{m}, \frac{\pi}{m}\right)$: let $\mathcal{N}_\theta^{(m)}$ be the number of eigenvalue angles in $\left[-\frac{\theta}{m}, \frac{\theta}{m}\right)$.

Theorem (E.M.–M. Meckes, 2016)

For $n, m \geq 1$,

$$d_{TV}(\mathcal{N}_\theta, \mathcal{N}_\theta^{(m)}) \leq \frac{2\sqrt{mn}\theta^2}{3\pi}.$$

Other neat stuff: the Riemann zeta function

Other neat stuff: the Riemann zeta function

Q: What to the eigenvalues of a random unitary matrix look like?

Other neat stuff: the Riemann zeta function

Q: What to the eigenvalues of a random unitary matrix look like?

A: Like the zeroes of the Riemann zeta function.

Other neat stuff: the Riemann zeta function

Q: What to the eigenvalues of a random unitary matrix look like?

A: Like the zeroes of the Riemann zeta function.

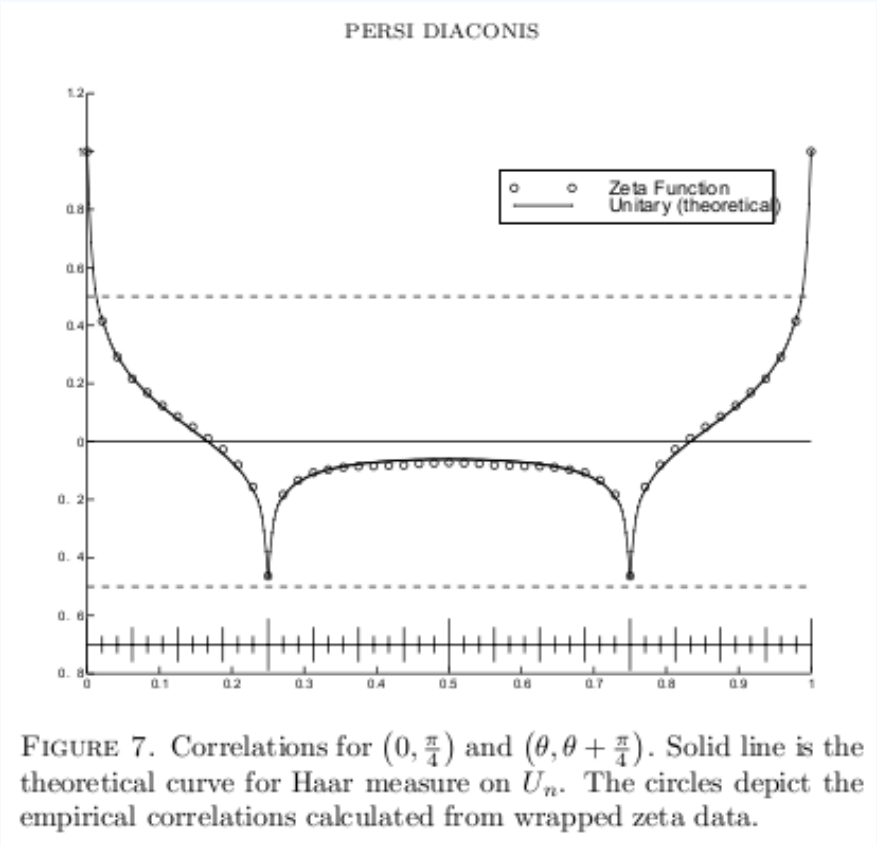


FIGURE 7. Correlations for $\left(0, \frac{\pi}{4}\right)$ and $\left(\theta, \theta + \frac{\pi}{4}\right)$. Solid line is the theoretical curve for Haar measure on U_n. The circles depict the empirical correlations calculated from wrapped zeta data.

Thank you.