1. (a) Find the general solution of \( \frac{dy}{dt} = 2ty. \)

This is a separable equation, so we separate variables:

\[
\int \frac{1}{y} \, dy = \int 2t \, dt
\]

\[
\ln |y| = t^2 + C
\]

\[
|y| = e^{t^2 + C} = e^C e^{t^2}
\]

\[
y = \pm e^C e^{t^2}
\]

Since \( y = 0 \) is also a solution,

\[
y = ke^{t^2} \] is the general solution.
(b) Verify that \( y(t) = (\sin t)e^{t^2} \) is a solution of
\[
\frac{dy}{dt} = 2ty + (\cos t)e^{t^2}.
\]

Just use the Rolle Poisson Principle:
\[
\frac{d}{dt} [ (\sin t)e^{t^2} ] = (\cos t)e^{t^2} + (\sin t)e^{t^2} \cdot 2t
\]
and \( 2ty + (\cos t)e^{t^2} = 2t(\sin t)e^{t^2} + \cos t)e^{t^2} \)

so \( \frac{dy}{dt} = 2ty + (\cos t)e^{t^2} \).

(c) Find the solution of the initial value problem
\[
\frac{dy}{dt} = 2ty + (\cos t)e^{t^2}, \quad y(0) = 4.
\]

This is a linear equation.
In part (a) we found the general solution of
the homogeneous equation \( \frac{dy}{dt} = 2ty \); \( y_h = ke^{t^2} \).
In (b) we saw that \( y_p = (\sin t)e^{t^2} \) is a
particular solution of the original equation.
So the general solution is \( y = (\sin t)e^{t^2} + ke^{t^2} \).
By the initial condition, \( 4 = 0 + k \cdot 1 \), so the
solution to the IVP is \( y = (\sin t + 4)e^{t^2} \).
2. Consider the modified logistic model

\[ \frac{dP}{dt} = P \left( 1 - \frac{P}{4} \right) \left( \frac{P}{2} - 1 \right). \]

(a) Find all the equilibrium solutions of this differential equation.

The RHS is 0 when \( P = 0, 4, 2 \), so these are the equilibria.

(b) Draw the phase line for this equation, and identify the equilibria as sinks, sources, or nodes.

\[ 5 \left( 1 - \frac{P}{4} \right) \left( \frac{P}{2} - 1 \right) < 0 \]

\[ 3 \left( 1 - \frac{P}{4} \right) \left( \frac{3}{2} - 1 \right) > 0 \]

\[ 1 \left( 1 - \frac{P}{4} \right) \left( \frac{1}{2} - 1 \right) < 0 \]

\[ -1 \left( 1 + \frac{4}{4} \right) \left( -\frac{1}{2} - 1 \right) > 0 \]

0 and 4 are sinks. 2 is a source.
(c) What will be the long-term behavior of the population in each of the following situations?

i. $P(0) = 1$

The population will decrease and approach 0.

ii. $P(0) = 3$

The population will increase and approach 4.

iii. $P(0) = 5$

The population will decrease and approach 4.

(d) What is the significance, for a population described by this model, of the population size 2?

A population larger than 2 (even only a little larger) will survive, whereas a population smaller than 2 (even only a little smaller) will not. So 2 is a threshold value, where the long-term fate of the population changes drastically.
3. (a) Verify that \( y_1(t) = \frac{1}{\sqrt{2t+1}} \) and \( y_2(t) = \frac{1}{\sqrt{2t+4}} \) are both solutions of

\[
\frac{dy}{dt} = -y^3.
\]

Use the Rat Poison Principle:

\[
\frac{dy_1}{dt} = \frac{d}{dt} (2t+1)^{-\frac{1}{2}} = -\frac{1}{2} (2t+1)^{-\frac{3}{2}} \cdot 2 = -\frac{1}{(2t+1)^{3/2}} = -y_1^3.
\]

and

\[
\frac{dy_2}{dt} = \frac{d}{dt} (2t+4)^{-\frac{1}{2}} = -\frac{1}{2} (2t+4)^{-\frac{3}{2}} \cdot 2 = -\frac{1}{(2t+4)^{3/2}} = -y_2^3.
\]

(b) Without finding the general solution of the differential equation, what can you say about solutions of \( \frac{dy}{dt} = -y^3 \) for which the initial condition \( y(0) \) satisfies \( \frac{1}{2} < y(0) < 1 \)?

\( y_1(0) = 1 \) and \( y_2(0) = \frac{1}{2} \), so \( y_2(0) < y(0) < y_1(0) \).

By the Uniqueness Theorem, solution curves don't cross, so \( y_2(t) < y(t) < y_1(t) \) for all \( t \).

Either from this or the phase line, we can see that \( y \rightarrow 0 \) as \( t \rightarrow \infty \).

Moreover, since \( y_2(t) \rightarrow \infty \) as \( t \rightarrow -2^+ \), we can tell that \( y \) becomes undefined for some \( t \geq -2 \).
4. (a) Use Euler's method with $\Delta t = 1$ to approximate the solution of the initial value problem \( \frac{dy}{dt} = (2 - y)(y + 1) \), \( y(0) = 1 \) at \( t = 3 \).

\[
t_{k+1} = t_k + \Delta t, \quad y_{k+1} = y_k + \Delta t \left( 2 - y_k \right) \left( y_k + 1 \right) \]

\[
t_0 = 0 \quad y_0 = 1
\]
\[
t_1 = 1 \quad y_1 = 1 + 1 \cdot (2 - 1)(1 + 1) = 3
\]
\[
t_2 = 2 \quad y_2 = 3 + 1 \cdot (2 - 3)(1 + 3) = -1
\]
\[
t_3 = 3 \quad y_3 = -1 + 1 \cdot (2 + 1)(-1 + 1) = -1
\]

(b) Does your approximation seem reasonable? Why or why not?

No, because it jumps back and forth across the equilibrium at 2 and settles at a different equilibrium.

In fact we know the correct solution approaches 2 (from the phase line).