1. Consider following two population models

\[ \frac{dx}{dt} = 2x - 1.2xy, \quad \frac{dy}{dt} = -y + 0.9xy, \]  
\[ \frac{dx}{dt} = 2x - 1.2xy, \quad \frac{dy}{dt} = y - 0.9xy. \]

(a) One of the models is a predator/prey system, and the other models two competing species. Which is which (explain your answer)?

Model (1) is predator/prey: the \(-1.2xy\) term in \(\frac{dx}{dt}\) means the presence of species y is bad for species x; the \(0.9xy\) term in \(\frac{dy}{dt}\) means the presence of x is good for y.

Model (2) is competing species: both \(xy\) terms are negative, so the presence of either species is bad for the other.

(b) For the predator/prey system, which variable represents the predators, and which represents the prey (explain)?

\(x\) is the prey (see above— the presence of y is bad for x) and \(y\) is the predator (the presence of x is good for y).
(c) Here are the direction fields for the two systems. Identify which direction field goes with which system.

Note: If \( x, y \) are both very large, then in system 1, \( \frac{dx}{dt} < 0 \) and \( \frac{dy}{dt} > 0 \). In 2, \( \frac{dx}{dt} < 0, \frac{dy}{dt} < 0 \).

(d) For each system, sketch a solution curve on the direction field corresponding to the initial condition \((2, \frac{3}{2})\). Describe the long-term behavior of the populations in both systems.

For system 1 (on the right above), the two species coexist in a periodic cycle. For system 2, the y's die out and the x's grow (x out-competes y).
2. Give the general solution to

\[
\begin{align*}
\frac{dx}{dt} &= 3x + y, \\
\frac{dy}{dt} &= -y.
\end{align*}
\]

\[
\frac{dy}{dt} = -y \Rightarrow y(t) = ke^{-t}
\]

\[
\Rightarrow \frac{dx}{dt} = 3x + ke^{-t}. \quad \text{(Non-homogeneous)}
\]

Solving the homogeneous part: \( \frac{dx}{dt} = 3x \Rightarrow x(t) = ke^{3t} \)

We guess a particular solution \( x_p(t) = ae^{-t} \).

Then \( x_p'(t) = -ae^{-t} \), and \( 3x_p(t) + ke^{-t} = (3a + k)e^{-t} \)

\(\Rightarrow\) we need \(-a = 3a + k \Rightarrow 0 = 4a + k\)

\(\Rightarrow\) \( a = -\frac{k}{4} \)

By the theorem on linear ODEs, this means that the general solution for \( \frac{dx}{dt} = 3x + ke^{-t} \)

is \( ke^{3t} - \frac{k}{4} e^{-t} \)

\(\Rightarrow\) General solution to the system: \[
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} = \begin{pmatrix}
ke^{3t} - \frac{k}{4} e^{-t} \\
ke^{-t}
\end{pmatrix}
\]
3. Solve the initial value problem

\[
\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0, \quad y(0) = 0, \quad y'(0) = 3.
\]

We try a solution of the form \( e^{st} = y(t) \):
\[
y'(t) = se^{st}, \quad y''(t) = s^2e^{st}.
\]
So for \( y \) to be a solution, we need:

\[
e^{st} [s^2 + 7s + 10] = 0 \iff s^2 + 7s + 10 = 0.
\]
But \( s^2 + 7s + 10 = (s + 5)(s + 2) \). So \( y(t) = e^{-2t} \) and \( y_1(t) = e^{-5t} \) are both solutions. Unfortunately, neither satisfies the initial conditions: both have \( y(0) = 1 \), and \( y_1(0) = -2 \), \( y_2(0) = -5 \).

So we try combining them: \( y(t) = k_1 e^{-2t} + k_2 e^{-5t} \).

Then \( y'(t) = -2k_1 e^{-2t} - 5k_2 e^{-5t} \), \( y''(t) = 4k_1 e^{-2t} + 25k_2 e^{-5t} \),

So \( y'' + 7y' + 10y = e^{-2t} [4k_1 - 14k_1 + 10k_1] + e^{-5t} [25k_2 - 35k_2 + 10k_2] \).

Also, \( y(0) = k_1 + k_2 \) and \( y'(0) = -2k_1 - 5k_2 \).

Solving \( k_1 + k_2 = 0 \) \( \Rightarrow -3k_2 = 3 \Rightarrow k_2 = -1 \Rightarrow k_1 = 1 \).

\[
\text{So } y(t) = e^{-2t} - e^{-5t}.
\]
4. Usually in zombie movies, zombies do not stop infecting new victims until they are destroyed by a human; humans destroy as many zombies as they can. This leads us to the following variation of the SIR model (where $H$ is the fraction of the initial population made of humans, $Z$ is the fraction made of zombies, and $D = 1 - H - Z$ is the fraction of dead zombies, which we need not include explicitly):

\[
\frac{dH}{dt} = -\alpha HZ, \\
\frac{dZ}{dt} = \alpha HZ - \gamma H.
\]

(a) Calculate the equilibrium points of the model.

\[
\frac{dH}{dt} = 0 \Rightarrow \alpha HZ = 0 \Rightarrow \frac{dZ}{dt} = -\gamma H
\]

\[
\Rightarrow \text{for } \frac{dH}{dt} \text{ and } \frac{dZ}{dt} = 0, \text{ we need } H = 0.
\]

(And whenever $H = 0$, both $\frac{dH}{dt}$ and $\frac{dZ}{dt} = 0$.)

Equilibrium solutions are

\[(H, Z) \text{ for any } 0 \leq Z \leq 1.\]

\[(0, Z_0)\]

(b) Find the region of the phase plane where $\frac{dZ}{dt} > 0$.

For $\frac{dZ}{dt} = H(\alpha Z - \gamma) > 0$, we need

\[
\alpha Z - \gamma > 0 \Rightarrow Z > \frac{\gamma}{\alpha}.
\]

(We assume that $H, Z > 0$.)
(c) Suppose that $\frac{x}{x} < 1$. Sketch the part of the phase portrait of the system where $H$ and $Z$ are positive. What does the model predict will happen to the human/zombie population?

\[ \frac{dZ}{dt} = \frac{y}{x}, \]
\[ \text{then } \frac{dH}{dt} = 0 \]
\[ \text{and } \frac{dH}{dt} < 0. \]

In fact, \[ \frac{dH}{dt} < 0 \text{ always.} \]

We have
\[ \frac{dZ}{dt} > 0 \text{ if } Z > \frac{y}{x} \]
and
\[ \frac{dZ}{dt} < 0 \text{ if } Z < \frac{y}{x}. \]

The model predicts that $\frac{y}{x}$ is a critical value for $Z$:

- if the initial zombie population is $\geq \frac{y}{x}$, all the humans die,
- but if the initial zombie population is $< \frac{y}{x}$, all the zombies are killed by humans.