Name: Solutions

Math 224 Exam 4
March 30, 2015

1. Consider the system \( \frac{dY}{dt} = AY \), where \( A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \).

   (a) Find the eigenvalues of \( A \).
   \[
   \text{Tr}(A) = 2, \quad \text{Det}(A) = -3 + 4 = 1 \implies \text{Char. poly: } \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2
   \]
   \[
   \implies \text{There is one eigenvalue: } \lambda = 1.
   \]

   (b) Find the eigenvectors of \( A \).
   \[
   A - \lambda I = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}.
   \text{Solving } \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
   \]
   \[
   \text{take } \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ as an eigenvector.}
   \]
   \[
   \text{Any other eigenvector is a scalar multiple of this one.}
   \]

   (c) Find the general solution of the system.
   \[
   \text{We know that if } A \text{ has a repeated ev and only one (up to scalar multiplication) ev, then the general solution is}
   \]
   \[
   Y(t) = e^{t} V_0 + t e^{t} V_1, \quad \text{where } V_1 = (A - \lambda I) V_0
   \]
   \[
   = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} V_0.
   \]
(d) Find the solution of the system with the initial condition $Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$Y(0) = v_0$, $V_1 = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$Y(t) = e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(e) Sketch the phase portrait, including the solution curve with the initial condition $Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$Y'(0) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Note: As $t \to -\infty$, $Y(t) \approx t e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for negative $Y(t)$ as above; this shows why the solution curves around as $t \to -\infty$. 
2. Consider the one-parameter family of linear systems given by

\[ \frac{dY}{dt} = \begin{pmatrix} a & a + 1 \\ a - 1 & a \end{pmatrix} Y. \]

(a) Sketch the corresponding curve in the trace-determinant plane.

\[ \text{Trace} = 2a \quad \text{Determinant} = a^2 - (a+1)(a-1) = a^2 - [a^2 - 1] = 1 \]

(b) Identify which types of behaviors the system exhibits for which values of \( a \).

Note that since \( T = 2a \), the line \( D = 1 \) intersects the \( D \)-axis when \( a = 0 \). Moreover, \( T^2 = 4D \Rightarrow 4a^2 = 4 \Rightarrow a = \pm 1 \) and \( T > 0 \) iff \( a > 0 \), so the intersection points of \( D = 1 \) with the parabola are at \( a = \pm 1 \), as labeled above.

- When \( a < -1 \), the system has a sink at 0.
- When \( -1 < a < 0 \), the system has a spiral sink at 0.
- When \( 0 < a < 1 \), the system has a spiral source at 0.
- When \( a = 1 \), the system has a source at 0.
- When \( a = 0 \), the system has a center at 0.

If \( a = -1 \), the matrix is \( \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \), which has only one eigenvector, so we have a pseudo-spiral sink: \( \quad \)

If \( a = 1 \), the matrix is \( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \), which likewise has only one eigenvector: we have a pseudo-spiral source.
3. Suppose a block with mass 1 is attached to the end of a spring with spring constant 5. The block is subject to a damping force proportional to its velocity, with a damping coefficient 4. Finally, an external time-dependent force of $\cos 2t$ acts on the block.

(a) Write a differential equation which models the behavior of the block.

$$\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 5y = \cos (2t)$$

(b) Find the general solution of your differential equation.

Homogeneous part: $y'' + 4y' + 5y = 0$, which has
char. poly. $\lambda^2 + 4\lambda + 5$, so e ws $\lambda = -2 \pm \sqrt{16 - 20} = -2 \pm i$

$\Rightarrow y_{gen, h}(t) = k_1 e^{-2t} \cos(t) + k_2 e^{-2t} \sin(t)$.

For the particular solution, consider

$y'' + 4y' + 5y = e^{2it}$

and guess $y_c(t) = \alpha e^{2it}$ (so $y'_c(t) = 2i \alpha e^{2it}$, $y''_c(t) = -4 \alpha e^{2it}$)

Need: $\alpha e^{2it} \left[-4 + 8i + 5\right] = \alpha e^{2it} \Rightarrow \alpha \left(1 + 8i\right) = 1$

$\Rightarrow \alpha = \frac{1}{1 + 8i} = \frac{1 - 8i}{65}$

$\Rightarrow y_c(t) = \left(\frac{1 - 8i}{65}\right)(\cos(2t) + i \sin(2t)) = \frac{1}{65} \left[\cos(2t) + 8 \sin(2t)\right] + \frac{i}{65} \left[\sin(2t) - 8 \cos(2t)\right]$

Since we are forcing with $\cos(2t)$, we need the real part:

$y_p(t) = \frac{1}{65} \left[\cos(2t) + 8 \sin(2t)\right]$ and so

$y_{gen}(t) = k_1 e^{-2t} \cos(t) + k_2 e^{-2t} \sin(t) + \frac{1}{65} \left(\cos(2t) + 8 \sin(2t)\right)$
(c) Describe the long-term behavior of the block.

In the long term, the solution to the homogeneous equation dies out and there is steady-state oscillation (corresponding to $y_p$) with period $\pi$.

(d) Suppose that at time 0 the block is at rest and the spring is stretched so that the block is a distance 1 from its equilibrium position. Determine the position of the block for all times $t$.

\[ y(0) = 1, \quad y'(0) = 0 \]

\[ y(t) = k_1 e^{-2t} \cos(t) + k_2 e^{-2t} \sin(t) + \frac{1}{65} (\cos(2t) + 8 \sin(2t)) \]

\[ y'(t) = -2k_1 e^{-2t} \cos(t) - k_2 e^{-2t} \sin(t) -2k_2 e^{-2t} \sin(t) + k_2 e^{-2t} \cos(t) + \frac{2}{65} (-\sin(2t) + 8 \cos(2t)) \]

\[ \Rightarrow \quad y(0) = k_1 + \frac{1}{65} = 1 \]

\[ y'(0) = -2k_1 + k_2 + \frac{16}{65} = 0 \]

\[ \Rightarrow \quad k_2 = 2k_1 \left( \frac{16}{65} \right) = \frac{128 - 14}{65} = \frac{112}{65} \]

\[ \Rightarrow \quad k_1 = \frac{64}{65} \Rightarrow k_2 = \frac{112}{65} \]

\[ y(t) = \frac{64}{65} e^{-2t} \cos(t) + \frac{112}{65} e^{-2t} \sin(t) + \frac{1}{65} (\cos(2t) + 8 \sin(2t)) \]