1. Show that the operator norm \( \|T\| \) of a linear map \( T \) is the smallest constant \( C \) such that

\[
\|Tv\| \leq C\|v\|
\]

for all \( v \in V \).

2. Prove that none of the following norms is associated to any inner product:

   (a) The \( \ell^1 \) norm on \( \mathbb{R}^2 \).
   (b) The supremum norm on \( C([0, 1]) \).
   (c) The operator norm on \( M_2(\mathbb{R}) \).

3. Prove that if \( A \in M_{m,n}(\mathbb{R}) \) has rank 1, then \( \|A\| = \|A\|_F \).

   *Hint: As you saw on a previous homework, if \( A \) has rank 1, then \( A = vw^T \) for some nonzero vectors \( v \in \mathbb{R}^m \) and \( w \in \mathbb{R}^n \). You can compute \( Ax \) and \( \|A\|_F \) explicitly in terms of \( v \) and \( w \).

   Then show that \( \|A\| \leq \|A\|_F \) by using the Cauchy–Schwarz inequality to show that \( \|Ax\| \leq \|A\|_F \|x\| \) for every \( x \in \mathbb{R}^n \).

   Finally, show that \( \|A\| \geq \|A\|_F \) by finding a specific unit vector \( x \in \mathbb{R}^n \) with \( \|Ax\| \geq \|A\|_F \).