Math 307 Homework
October 30, 2015

1. Suppose that $\mathbf{U} \in \mathrm{M}_n(\mathbb{C})$ is unitary.

   (a) Compute $\|\mathbf{U}\|$.

   (b) Compute $\|\mathbf{U}\|_F$.

2. Suppose that $\mathbf{U} \in \mathrm{M}_m(\mathbb{C})$ and $\mathbf{V} \in \mathrm{M}_n(C)$ are unitary, and $\mathbf{A} \in \mathrm{M}_{m,n}(\mathbb{C})$. Prove that

$$\|\mathbf{UAV}\| = \|\mathbf{A}\| \qquad \text{and} \qquad \|\mathbf{UAV}\|_F = \|\mathbf{A}\|_F .$$

3. Find the QR decomposition of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

4. The $n \times n$ **DFT matrix** (for **discrete Fourier transform**) is the matrix $\mathbf{F} \in \mathrm{M}_n(\mathbb{C})$ with entries

$$f_{jk} = \frac{1}{\sqrt{n}} \omega^{jk},$$

   where $\omega = e^{2\pi i/n} = \cos(2\pi/n) + i \sin(2\pi/n)$. Prove that $\mathbf{F}$ is unitary.

   Hints: If you're not accustomed to complex exponentials, here's what you need to know about them for this problem:

   - If $x, y \in \mathbb{R}$ then $e^{i(x+y)} = e^{ix} e^{iy}$. In particular, $e^{ikx} = (e^{ix})^k$ if k is a positive integer.
   - $e^{2\pi i} = 1$, so here $\omega^n = 1$, but $e^{2\pi i k/n} \neq 1$ for $k = 1, \dots, n-1$.
   - If $x \in \mathbb{R}$, then $\overline{e^{ix}} = e^{-ix}$.

   The following identity is also useful here:

$$\sum_{k=0}^{n-1} z^k = \frac{1 - z^n}{1 - z}$$

   for any $z \in \mathbb{C}$ with $z \neq 1$. (This can be proved by simply multiplying both sides by $1 - z$.)