Math 307 Homework
October 5, 2015

1. Prove that if $\mathbf{A} \in \mathrm{M}_{m,n}(\mathbb{F})$ has rank $r \geq 1$, then there exist $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{F}^m$ and $\mathbf{w}_1, \ldots, \mathbf{w}_r \in \mathbb{F}^n$ such that

$$\mathbf{A} = \sum_{i=1}^{r} \mathbf{v}_i \mathbf{w}_i^{\mathrm{T}}.$$

2. (a) Prove that if $\mathbf{A}, \mathbf{B} \in \mathrm{M}_{m,n}(\mathbb{F})$, then

$$\operatorname{rank}(\mathbf{A} + \mathbf{B}) \leq \operatorname{rank} \mathbf{A} + \operatorname{rank} \mathbf{B}.$$

   (b) Prove that if

$$\mathbf{A} = \sum_{i=1}^{r} \mathbf{v}_i \mathbf{w}_i^{\mathrm{T}}$$

   for some $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{F}^m$ and $\mathbf{w}_1, \ldots, \mathbf{w}_r \in \mathbb{F}^n$, then $\operatorname{rank} \mathbf{A} \leq r$.

3. Give another proof of Theorem 2.29 using the two exercises above.

4. Suppose that, for a given $\mathbf{A} \in \mathrm{M}_3(\mathbb{R})$, there is a plane P in $\mathbb{R}^3$ such that the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in P$. Prove that the set of solutions of the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ is a line through the origin in $\mathbb{R}^3$.