Math 307 Homework
September 18, 2015

1. Find the matrix of the linear map $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{R}^2$ which first reflects across the y-axis, then rotates counterclockwise by $\pi/4$ radians, then stretches by a factor of 2 in the y-direction.

2. Compute
$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}^{-1}.$$

3. An $n \times n$ matrix $\mathbf{A}$ is called **upper triangular** if $a_{ij} = 0$ whenever $i > j$.

   (a) Suppose that $\mathbf{A}, \mathbf{B} \in \mathrm{M}_n(\mathbb{F})$ are both upper triangular. Prove that $\mathbf{AB}$ is also upper triangular. What are the diagonal entries of $\mathbf{AB}$?

   Warning: Don't be tricked by this into thinking that $\mathbf{AB} = \mathbf{BA}$ for upper triangular matrices!

   (b) Suppose that $\mathbf{A} \in \mathrm{M}_n(\mathbb{F})$ is upper triangular and invertible. Prove that $\mathbf{A}^{-1}$ is also upper triangular.

   Hint: Think about the row operations used in computing $\mathbf{A}^{-1}$ via Gaussian elimination.

4. Suppose that $\mathbf{A} \in \mathrm{M}_{m,n}(\mathbb{F})$ is **right-invertible**, meaning that there is a $\mathbf{B} \in \mathrm{M}_{n,m}(\mathbb{F})$ such that $\mathbf{AB} = \mathbf{I}_m$. Show that $m \leq n$.

   Hint: Show that given any $\mathbf{b} \in \mathbb{F}^m$, the $m \times n$ linear system
$$\mathbf{Ax} = \mathbf{b}$$
   is consistent, and use Theorem 1.2.