Math 307 Homework
September 25, 2015

1. Two *nonzero* vectors $v, w \in V$ are called **collinear** if there is a scalar $c \in \mathbb{F}$ such that $v = cw$.

   (a) Given two nonzero vectors $v, w \in V$, prove that (v, w) is linearly dependent if and only if v and w are collinear.

   (b) Give an example of a list of three linearly dependent vectors such that no two are collinear.

2. Determine whether each of the following lists of vectors in $\mathbb{R}^3$ is linearly independent.

   - $\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right)$
   - $\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right)$

3. Suppose that $\mathbf{A} \in \mathrm{M}_n(\mathbb{F})$ is upper triangular and that all the diagonal entries are nonzero (i.e., $a_{ii} \neq 0$ for each $i = 1, \ldots, n$). Prove that $\ker \mathbf{A} = \{\mathbf{0}\}$.

4. Let $n \geq 1$ be an integer, and suppose that for every $x \in \mathbb{R}$,

$$\sum_{k=1}^{n} a_k \sin(kx) = 0$$

   where constants $a_1, \ldots, a_n \in \mathbb{R}$ are constants. Prove that $a_1 = \cdots = a_n = 0$.

   Hint: Consider the linear map $\boldsymbol{D}^2 : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $\boldsymbol{D}^2 f = f''$, where $C^\infty(\mathbb{R})$ is the space of infinitely differentiable functions on $\mathbb{R}$, and use Theorem 2.7.