In class, we defined a function $\mu : \mathcal{B} \times \Omega \to \mathbb{R}$ so that for each $r \in \mathbb{Q}$,

$$\mu((-\infty, r], \omega) = \mathbb{P}[X \leq r|\mathcal{G}]_\omega$$

with probability 1.

(a) Let $\mathcal{L}$ be the class of $\mathcal{B}$-sets H such that $\mu(H, \cdot)$ is $\mathcal{G}$-measurable. Show that $\mathcal{L}$ is a λ-system, and prove that this means that $\mu(H, \cdot)$ is $\mathcal{G}$-measurable for every $H \in \mathcal{B}$.

(b) Since $\mu((-\infty, r], \omega) = \mathbb{P}[X \leq r|\mathcal{G}]_\omega$, it follows that for each fixed $G \in \mathcal{G}$,

$$\int_G \mu(H, \omega) d\mathbb{P}(\omega) = \mathbb{P}[\{X \in H\} \cap G],$$

for every H of the form $(-\infty, r]$ for $r \in \mathbb{Q}$. Show that this means that the above equality holds for all $H \in \mathcal{B}$.