M445: Heat equation with sources

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I. On Fourier and Newton’s cooling laws

The Newton’s law claims the temperature rate to be proportional to the difference:

\[ \frac{dT}{dt} = -\alpha (T - T_0) \]  

(1)

The Fourier law postulates the heat-flux to be proportional to the temperature gradient:

\[ \frac{dQ}{dt} = -\int_{\Sigma} \kappa \nabla T \cdot N dS; \]

\[ Q = cT \]

with coefficients \( c = \) heat capacity; \( \kappa = \) heat conductivity. Two descriptions deal with different time scales: fast for the Fourier and slow for the Newton.

A physical model could be a fluid undergoing turbulent mixing as it cools down, e.g., buoyancy-driven convection in a pool with a freezing surface. Call temperatures \( T_0 \) (air) and \( T_b \) (initial water).

We consider a circulation pattern that randomly replaces surface parcels at a constant rate (so called renewal model). The fluid patches that come to the surface could have the ambient water temperature \( T > T_0 \), and thus capable of releasing heat. Or they’ve already participated in the heat-exchange on the previous time-step, which brought their temperature down to \( T \approx T_0 \), hence rendered them incapable of further cooling.

Let \( \phi (t) \) denotes a fraction of the surface exposed (and capable) of the heat exchange. A constant replacement rate makes \( \phi (t) \) an exponential function, \( \phi (t) = \frac{1}{\tau} e^{-t/\tau} \), where \( \tau \) is the slow time-scale.

We take the standard \( \text{erf} \) -solution of the half-space problem and its heat-flux

\[ T (z, t) = (T_0 - T_b) \text{erf} \left( \frac{z}{\sqrt{\kappa t}} \right) + T_b \]

\[ Q (0, t) = -k \left( \frac{\partial T}{\partial z} \right) |_{z=0} = -\frac{k}{\sqrt{\pi \kappa t}} (T_0 - T_b) \]

Here \( \kappa \) denotes the heat-diffusivity, and \( k \) -heat conductivity.
Averaging out the fast time scales we get

\[ T(z) = \int_0^\infty \phi(t) T(z,t) \, dt = (T_0 - T_b) e^{-z/\sqrt{\kappa \tau}} \]  

(3)

\[ Q(0) = \int_0^\infty \phi(t) Q(0,t) \, dt = -\frac{k}{\sqrt{\kappa \tau}} (T_0 - T_b) = -\frac{\kappa c_p \rho}{l_\theta} (T_0 - T_b) \]

where \( l_\theta \) is the length scale of the average temperature profile, \( c_p \)-specific heat at constant pressure and \( \rho \) -density.

The second line of (3) is essentially the Newton’s law.

Mathematically, the erf-solution (e.g. for a cooling bar \([-a, a]\))

\[ T(x,t) = \text{erf} \left( \frac{x + a}{\sqrt{t}} \right) - \text{erf} \left( \frac{x - a}{\sqrt{t}} \right) \approx \frac{2a}{\sqrt{t}} + O(t^{-3/2}) \]

has a polynomial fall-off in \( t \) (the same would hold for the space-average temperature over \([-a, a]\)). However, averaging over the slow (Newton) time scale \( \tau \), i.e. taking time-convolution of \( T \) with \( e^{-t/\tau} \), we get

\[
\int_0^t \frac{e^{-(t-s)/\tau}}{\sqrt{s}} ds \approx e^{-t/\tau} \left( c_0 + \frac{c_1}{\sqrt{t}} + \ldots \right)
\]

i.e. Newton’s exponential fall-off.

II. Heat equation with delta-sources

We write a typical heat-diffusion problem using symbolic operator notation

\[
\begin{align*}
    u_t + L[u] &= F \\
    u|_{t=0} &= f
\end{align*}
\]  

(4)

Here \( L \) could be an ordinary differential operator \(-\partial p \partial + q\) on \([0, l]\) with suitable boundary conditions at \(\{0, l\}\), or more general elliptic pde \(L = -\nabla \cdot p \nabla + q\), on region \(D \subset \mathbb{R}^n\) with boundary \(\Gamma\), and boundary condition \(B[u] = (a + b \partial_n) u|_{\Gamma}\).

The formal (ODE-type) solution of (4) is given by the \textit{operator-exponential}

\[ u = e^{-tL}[f] + \int_0^t e^{-(t-s)L} [F(s)] \, ds \]  

(5)

analogous to the matrix-exponential.
Such operator-exponential represents a fundamental solution of problem (4). One could show that operator $e^{-tL}$ acting on functions $\{f(x)\}$ is given by an integral kernel $G(x, \xi, t)$, called Green’s function of the problem,

$$G[f] = \int_D G(x, \xi, \ldots) f(\xi) \, d\xi$$

We are interested in the delta-source $F = h(t) \delta(x - x_0)$. If $G(x, \xi, t)$ denotes the Green’s function of $L - B$, then solution (5)

$$u(x, t) = \int_0^t G(x, x_0, t - s) h(s) \, ds$$

We evaluate (7) for the standard Gaussian $G = \frac{1}{(4\pi t)^{n/2}} e^{-x^2/4at}$ i.e. $L = -\alpha \Delta$ on $\mathbb{R}^n$

$$u(x, t) = \frac{1}{(4\pi \alpha t)^{n/2}} \int_0^t e^{-x^2/4as} s^{n/2} h(t - s) \, ds$$

and treat 2 cases.

A. Constant source $h = \text{Const}$

Here (8) yields after the change $s \to z = \frac{x^2}{4as}$

$$u = \frac{|x|^{2-n}}{4\alpha} \int_{x^2/4at}^\infty z^{n/2-2} e^{-z} \, dz = \frac{|x|^{2-n}}{4\alpha} \Gamma \left( \frac{n}{2} - 1, \frac{x^2}{4at} \right)$$

expressed in terms of incomplete Euler gamma function

$$\Gamma(\nu, p) = \int_p^\infty e^{-z} z^{\nu-1} \, dz$$

of order $\nu = \frac{n}{2} - 1$, depending on space-dimension. Dimensions $n = 1, 2, 3$ can be expanded for small $x$ as

$$\Gamma(-\frac{1}{2}, p) = \frac{2}{\sqrt{p}} - 2\sqrt{\pi} + 2\sqrt{p} - \frac{1}{3} p^{3/2} + \frac{1}{15} p^{5/2} + O(p^{7/2})$$

$$\Gamma(0, p) = \left(-\gamma + \ln \frac{1}{p}\right) + \frac{1}{4} p^2 + \frac{1}{18} p^3 + O(p^4)$$

$$\Gamma(\frac{1}{2}, p) = \sqrt{\pi} - 2\sqrt{p} + \frac{2}{3} p^{3/2} - \frac{1}{5} p^{5/2} + \frac{1}{21} p^{7/2} + O(p^4)$$
with Euler constant \( \gamma = .577 \). We plot all 3 gammas

![Graph of gamma functions](image)

Incomplete gamma-functions \( \Gamma (\nu, p) \) for \( \nu = -\frac{1}{2}, 0; \frac{1}{2} \)

and write the corresponding solutions \( u \) expanded in small \( p = \frac{x^2}{4\alpha t} \)

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<td>( \frac{1}{4\alpha} \left( -0.577 + \ln \frac{4\alpha t}{x^2} + \frac{1}{4} \alpha t^2 - \frac{1}{64 \alpha \sqrt{\pi} t^2} + \ldots \right) )</td>
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Notice that in 1D solution has an asymptotic limit as \( t \to \infty \)

\[
u(x,t) \simeq \sqrt{\frac{t}{\alpha}} - \frac{\sqrt{\pi} |x|}{2 \alpha} \]

whereas 3D-one converges to a potential-type equilibrium

\[ u(x,t) \to \frac{\sqrt{\pi}}{4\alpha |x|} \]

The exact solutions for \( n = 1, 2, 3 \)

\[
u = \frac{|x|}{4\alpha} \Gamma \left( -\frac{1}{2}, \frac{x^2}{4\alpha t} \right) \]
\[
u = \frac{1}{4\alpha} \Gamma \left( 0, \frac{x^2}{4\alpha t} \right) \]
\[
u = \frac{1}{4\alpha |x|} \Gamma \left( \frac{1}{2}, \frac{x^2}{4\alpha t} \right) \]
are plotted below as radial profiles $u(r,t)$ in 3D-space-time view

1D

2D

3D

Temperature profile for 1D, 2D and 3D steady point sources
We also show their time snapshots

\[ \begin{align*}
1D & \quad 2D \\
3D
\end{align*} \]

*Time slices of temperature profiles in 1, 2 and 3D*

A typical pattern shows accumulation of heat near the source and its spread outward. The rate of accumulation depends on space dimension and steepens with the increase of \( n \).

**B. Time periodic source:** \( h = \cos \omega t \)

Here solution

\[ u = \int_0^t \cos \omega (t-s) \frac{e^{-x^2/4\alpha s}}{(4\pi \alpha s)^{n/2}} \, ds \]

The integral has no closed form expression in known (elementary or special) functions. But its large-time asymptotics could be reduced to Fourier transforms of function:

\( f(t) = \frac{e^{-x^2/4\alpha t}}{(4\pi \alpha t)^{n/2}} \). Namely,

\[ u \approx \cos \omega t \left( \int_0^\infty \cos \omega s f(s) \, ds \right) - \sin \omega t \left( \int_0^\infty \sin \omega s f(s) \, ds \right) \]

\[ \hat{f}_c(\omega) \quad \hat{f}_s(\omega) \]
The complete (half-line Fourier) transform of $f$ is expressed through the modified Bessel (Kelvin) function $K_{n/2-1}$

$$\hat{f}(\omega) = \int_0^\infty e^{i\omega t} e^{-x^2/t} \frac{dt}{t^{n/2}} = 2e^{i\pi(\frac{n-2}{2})} \left( \frac{\sqrt{\omega}}{|x|} \right)^{n/2-1} K_{n/2-1} \left( 2\sqrt{-i\omega} |x| \right) \quad (9)$$

In special cases, e.g. 1D $K_{-1/2}$ is an elementary function, so integral (9) is simplified to

$$\hat{f} = \frac{1 + i}{\sqrt{2\omega}} e^{-\sqrt{2\omega}|x|} \left( \cos \sqrt{2\omega} |x| + i \sin \sqrt{2\omega} |x| \right)$$

We have thus shown that the asymptotic pattern consists of exponentially attenuated propagating heat-waves

$$u \approx e^{-\sqrt{2\omega}|x|} \cos \left( \sqrt{2\omega} |x| \pm \omega t \right)$$

Let us remark that the relation between the wave number $k = \sqrt{\omega}$ and frequency $\omega$ is consistent with the heat-diffusion dispersion law: $i\omega = k^2$.

III. Equilibria for heat-diffusion problems

We use operator formalism (5) for a typical heat-diffusion problem (4) to write its formal solution in terms of operator exponentials, analogous to the matrix-exponential. All functions $u, f, F$ could be expanded in terms of eigenfunctions $\{\psi_k\}$ of operator $L$, (rather eigenmodes of the boundary value problem $L; B$)

$$L[\psi_k] = \lambda_k \psi_k$$
$$B \psi_k \big|_\Gamma = 0 \quad (10)$$

In particular, Green’s function is expanded as

$$G(x, \xi, t) = \sum_k e^{-\lambda_k t} \frac{\psi_k(x) \bar{\psi}_k(\xi)}{||\psi_k||^2}$$
and solution (11) becomes

\[ u(x,t) = \sum_k \left( \hat{f}_k e^{-\lambda_k t} + \int_0^t e^{-(t-s)\lambda_k} \left[ \hat{F}_k(s) \right] ds \right) \psi_k(x). \]  

Here \( \{ \hat{f}_k \} ; \{ \hat{F}_k(t) \} \) denote generalized Fourier coefficients of \( f \) and \( F \)

\[
\hat{f}_k = \frac{\langle f(x) \psi_k \rangle}{\|\psi_k\|^2}
\]

in the sense of \( L^2 \) (square-mean) inner product.

A simple equilibrium solution \( v \) of problem (4) with a stationary (time-independent) source \( F \) is given by

\[ L[v] = F \Rightarrow v = L^{-1}[F] \]  

By analogy with exponential \( e^{-tL} \) operator \( L^{-1} \) could be represented by an integral kernel (Green’s function)

\[ K(x,\xi) = \sum_k \frac{1}{\lambda_k} \frac{\psi_k(x) \bar{\psi}_k(\xi)}{\|\psi_k\|^2} \]

expanded in eigenmodes of \( L \). Hence

\[ v(x) = \sum_k \frac{\hat{F}_k}{\lambda_k} \psi_k(x) \]

The latter is easily shown to be a limit of solution (11) as \( t \to \infty \), provided all eigenvalues \( \{\lambda_k\} \) of \( L \) are positive. Indeed, convolution integral (11) becomes

\[ u = \frac{I - e^{-tL}}{L} [F] + e^{-tL} [f] \to L^{-1}[F] = v, \text{ as } t \to \infty \]

A. Periodic equilibria

More interesting case arises for a periodic source \( F(x,t) \). One asks the same two questions as above

1. whether periodic solutions \( v \) exist for (4)
2. whether they are stable, in the sense that any \( u(x,t) \to v \) as \( t \to \infty \)

Both are easily answered using the above operator (ODE)-formalism.

We first consider a single frequency case \( F = F(x) e^{i\omega t} \) in the complex form,

\[
\begin{cases}
\quad u_t + L[u] = F e^{i\omega t} \\
\quad u|_{t=0} = f
\end{cases}
\]
Formal solution of IVP (13)

\[
    u = \frac{e^{i\omega t} - e^{-Lt}}{i\omega + L} [F] + e^{-Lt} [f] \\
    = e^{i\omega t} \left( \frac{1}{i\omega + L} \right) [F] + e^{-Lt} \left[ f - \left( \frac{1}{i\omega + L} \right) F \right]
\]

is decomposed into the periodic component \( v (x) e^{i\omega t} \), where equilibrium \( v \) satisfies

\[
    (i\omega + L) v = F \Rightarrow v = (i\omega + L)^{-1} F
\]

and negative exponential \( e^{-Lt} [\cdots] \). As above operators \((i\omega + L)^{-1} e^{-Lt} \) are given by (complex-valued) Green’s functions \( K (x, \xi; i\omega) \); \( G (x, \xi; t) \), or else could be expanded in eigenmodes

\[
    v (x) = \sum_k \frac{\hat{F}_k}{i\omega + \lambda_k} \psi_k (x)
\]

From complex form (16) one could easily get the real periodic solution

\[
    \begin{cases}
        u_t + L [u] = F \cos \omega t \\
        u|_0 = f
    \end{cases}
\]

by taking the real and imaginary parts of (14)

\[
    \text{Re} \left( \frac{e^{i\omega t}}{i\omega + L} \right) = \frac{L}{L^2 + \omega^2} \cos \omega t + \frac{\omega}{L^2 + \omega^2} \sin \omega t
\]

This yields the IVP-solution (17) written as

\[
    u = \left( \cos \omega t \frac{L}{L^2 + \omega^2} + \sin \omega t \frac{\omega}{L^2 + \omega^2} \right) F + e^{-Lt} \left( f - \frac{L}{L^2 + \omega^2} F \right)
\]

in the operator-form, or an equivalent series expansion

\[
    u = \sum_k \left\{ \frac{\lambda_k \cos \omega t + \omega \sin \omega t}{\lambda_k^2 + \omega^2} \hat{F}_k \\
    + e^{-\lambda_k t} \left( \hat{f}_k - \frac{\lambda_k}{\lambda_k^2 + \omega^2} \hat{F}_k \right) \right\} \psi_k (x)
\]

The latter clearly demonstrates that \( u (x, t) \) converges to a periodic equilibrium \( v (x, t) = \text{Re} (v (x) e^{i\omega t}) \), provided all eigenvalues \( \lambda_k \) are positive, so exponential terms drop in (14)-(18).
B. Multiple frequency case

Here $F$ is represented by a time-Fourier series

$$F = \sum_{m} e^{i\omega_m t} F_m(x)$$

In the periodic case all frequencies are multiples of a single (lowest) one $\omega_m = m\omega$ and the period of $F$ is $T = \frac{2\pi}{\omega}$. More generally, $\{\omega_m\}$ are arbitrary real numbers, the so called \textit{frequency spectrum} of $F$.

We seek partial (periodic) solution $v$ in the form

$$v = \sum_{m} e^{i\omega_m t} v_m(x)$$

(19)

with undetermined Fourier coefficients $\{v_m\}$. The substitution in (4) determines each one of them via (15)

$$v_m = (i\omega_m + L)^{-1} F_m$$

So $v$ is expanded in the time Fourier series (19) with the same period (or quasiperiods) as $F$.

An interesting example of multiple frequencies arises for a periodically moving point-source

$$F = \delta(x - a \cos \omega t)$$

(20)

We consider it on a symmetric interval $[-l,l]$ with amplitude of oscillation $a < l$. Generalized time-periodic function (20) has a frequency Fourier expansion $F = \sum_{m} e^{im\omega t} F_m(x)$ with coefficients

$$F_m(x) = \frac{\cos\left(m \cos^{-1}\left(\frac{x}{a}\right)\right)}{\pi \sqrt{a^2 - x^2}} = \frac{T_m\left(\frac{x}{a}\right)}{\pi \sqrt{a^2 - x^2}}$$

whose numerators are made of the classical Tchebyshev polynomials of the first kind. We plot a few of them

\footnote{Function $F$ is called \textit{quasi-periodic} if its spectrum is made of linear combinations of a finite (basic) set $\{\omega_1; \ldots; \omega_p\}$

$$\omega_m = \sum_{k=1}^{p} n_k \omega_k$$

with integer coefficients $n_k$. Otherwise, it is called almost periodic.}
As a consequence we get the periodic equilibrium for the moving-source problem, expanded in the double series

\[ v(x, t) = \frac{1}{\pi} \sum_{m=0}^{\infty} \sum_{\lambda_k} \left( \frac{\lambda_k \cos m \omega t + m \omega \sin m \omega t}{\lambda_k^2 + (m \omega)^2} \right) \]

\[ \times \frac{\langle T_m \left( \frac{x}{a} \right) / \sqrt{a^2 - x^2} \psi_k \rangle}{\| \psi_k \|^2} \psi_k(x) \] 

assuming all eigenvalues of \( L \) positive.

**Problems:**

1. Specify expansion (21) for the Dirichlet and Neumann problem on \([-l, l]\), \( L = -\partial^2 \).

2. Compute the first 5 frequency modes \( m = 0, ..., 5 \).

3. Plot approximate periodic equilibrium (21) by truncating both series. Use Mathematica!