Quantized Neumann problem, separable potentials on $S^n$ and the Lamè equation

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The paper studies spectral theory of Schrödinger operators $H = h^2 \Delta + V$ on the sphere from the standpoint of integrability and separation. Our goal is to uncover the fine structure of spec $H$, i.e., asymptotics of eigenvalues and spectral clusters, determine their relation to the underlying geometry and classical dynamics and apply this data to the inverse spectral problem on the sphere. The prototype model is the celebrated Neumann Hamiltonian $p^2 + V$ with quadratic potential $V$ on $S^n$. We show that the quantum Neumann Hamiltonian (Schrödinger operator $H$) remains an integrable and find an explicit set of commuting integrals. We also exhibit large classes of separable potentials $\{V\}$ based on ellipsoidal coordinates on $S^n$. Several approaches to spectral theory of such Hamiltonians are outlined. The semiclassical problem (small $\hbar$) involves the $EKB(M)$-quantization of the classical Neumann flow along with its invariant tori, Maslov indices, etc., all made explicit via separation of variables. Another approach exploits Stäckel–Robertson separation of the quantum Hamiltonian and reduction to certain ODE problems: the Hill’s and the generalized Lamè equations. The detailed analysis is carried out for $S^2$, where the ODE becomes the perturbed classical Lamè equation and the Schrödinger eigenvalues are expressed through the Lamè eigendata. © 1995 American Institute of Physics.

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The classical Neumann problem represents a harmonic oscillator restricted on the unit sphere \( S^n = \{ x : |x| = 1 \} \) in \( \mathbb{R}^{n+1} \), or a spherical pendulum in the field of linear force \( F = Ax \). Its Hamiltonian \( h = \frac{1}{2}(p^2 + V) \) is made of the standard kinetic energy \( T = \frac{1}{2}p^2 \), the Euclidean metric-tensor on \( T^*(S^n) \subset \mathbb{R}^{2n+2} \) and quadratic potential \( V = Ax \cdot x \) with diagonal matrix \( A = \text{diag}(\alpha_1; \ldots; \alpha_{n+1}) \).

The classical Neumann problem is well known to be integrable\(^1\) the commuting integrals being

\[
 f_k(x;p) = x_k^2 + \sum_{i \neq k} \frac{J_{ik}^2}{\alpha_k - \alpha_i}; k = 1; \ldots n
\]

(see Ref. 2). Here \( \{ J_{ij} = x_ip_j - x_jp_i \} \) denote the components of the angular momentum \( J = x \wedge p \).

One can show that Hamiltonians \( \{ f_k \} \) Poisson commute on the phase-space \( \mathbb{R}^{2n+2} \). Furthermore, they obey the relations,

\[
 \sum f_i(x;p) = x^2,
\]

while their weighted combination with weights \( \{ \alpha_i \} \) gives the Hamiltonian itself

\[
 h = \sum \alpha_i f_i(x;p) = \sum \alpha_i x_i^2 + \sum J_{ij}^2 = V(x) + |J|^2.
\]

These relations allow one to reduce the Neumann system from the phase-space \( \mathbb{R}^{2n+2} \) to a cotangent bundle \( T^*(S^n) \subset \mathbb{R}^{2n+2} \), defined by two constraints: \( x^2 = 1 \) and \( x \cdot p = 0 \). The reduced system remains integrable. The latter is far from obvious, as there are no general reasons for a constrained system to remain integrable. Indeed, the trivial integrable Hamiltonian \( p^2 \) on \( \mathbb{R}^n \) restricted on a hypersurface \( \Sigma \) gives rise to a geodesic flow, which would be typically chaotic rather than integrable.

In Sec. II we shall see that classical Hamiltonian \( h \), along with conserved integrals \( \{ f_i \} \), could be naturally quantized to produce a Schrödinger operator

\[
 H = -\frac{i}{\hbar} \Delta + V
\]

with potential \( V = \frac{1}{2}Ax \cdot x \) on \( S^n \).

In the spectral theory of Schrödinger operators one is interested in eigenvalues \( \{ \lambda \} \) of \( H_\lambda \) and their connection to geometry (potential, metric, etc.). Such geometric data encoded in \( \lambda \)'s could eventually lead to solution of the inverse problem and characterization of isospectral classes of \( \{ V \} \). The \( n \)-sphere Schrödinger theory provides an interesting example of the inverse problem, that was subject of many works (see references in Ref. 3). Though a great deal is known about spectra of Schrödinger operators on \( S^n \), we are still far from resolving such basic problems, like rigidity (whether the isospectral class consists of all rotations of \( V \) by the group \( \text{SO}(n+1) \)), or the inverse spectral problem. Both results however, were established in special cases.\(^4\) \(^6\) An essential feature of the \( n \)-sphere Schrödinger theory is clustering of eigenvalues of \( H_\lambda \) about the unperturbed and
highly degenerate spectrum of the Laplacian \( \text{spec}(\Delta) = \{ \lambda_i = l(l + n - 2) \} \). Each cluster \( \Lambda_i \) consists of spectral shifts \( \{ \mu_{im} \} \), localized near unperturbed \( \lambda_i \), that result from breaking the underlying \( \text{SO}(n + 1) \)-symmetry of \( \Delta \) by perturbation \( V \),

\[
\Lambda_i = \{ \lambda_i + \mu_{im} : m = 1; \ldots; d_i \},
\]

Here \( d_i \) denotes the multiplicity of the free (Laplace) eigenvalue \( \lambda_i \), i.e., dimension of the degree \( l \) spherical harmonics on \( \mathbb{R}^{n+1} \). Clusters \( \Lambda_i \) are well separated, \( \text{dist}(\Lambda_i; \Lambda_{i+1}) = \mathcal{O}(l) \), while their size remains bounded or decreases with \( l \).

\[
|\Lambda_i| = \begin{cases} 
\mathcal{O}(1), & \text{for even/generic } V \\
\mathcal{O}(l^{-2}), & \text{for odd } V 
\end{cases}
\]

The large-\( l \) asymptotics of spectral shifts \( \{ \mu_{im} \} \) could be described by cluster-distribution measures \( \{ d\nu_i \} \). It turns out that the \( l \)-asymptotics of measures \( \{ d\nu_i \} \) can be expressed through the Radon and related transforms of potential \( V \). Furthermore, in case of zonal (axisymmetric) potentials \( \{ V \} \) asymptotic expansion was established for individual spectral shifts \( \{ \mu_{lm} \} \). The argument of Ref. 4 was based on the auxiliary \( \text{SO}(n) \)-symmetry of zonal potentials and the corresponding commuting integrals (like angular momentum \( J_z = xp_y - yp_x \) on \( S^2 \)), that allows one to “quantize” the joint \( \{ H_Y; J_z \} \)-spectrum by pairs of integers \( \{(l;m): -l \leq m \leq l\} \). Then the \( ml \)-th spectral shift was shown to be

\[
\mu_{lm} = a \left( \frac{m}{l} \right) + \frac{1}{l} b \left( \frac{m}{l} \right) + \frac{1}{l^2} c \left( \frac{m}{l} \right) + \cdots \quad \text{as } l \to \infty.
\]

Two quantum numbers acquire a transparent meaning here with \( l \) measuring the principal cluster-number and \( m \) standing for the quantized angular momentum \( J_z \) for the quantized angular momentum \( J_z \). Coefficients \( \{a(x); b(x); c(x)\} \) are certain functions on \([0; 1]\), depending on \( V \). Precisely, \( a \) is a “Radon transform of \( V \)” (i.e., \( V \) integrated along great circles \( \gamma \)), while \( b \) and \( c \) involve more complicated polynomial expressions in \( V \) and its derivatives integrated along \( \gamma \).4,6

Semiclassical expansion (2) could be also viewed as resulting from the effective (“averaged”) perturbation \( V_{\text{eff}} = \sqrt{\Delta} \). In other words, operators \( H = -\Delta + V \) and \( H_{\text{eff}} = -\Delta + V_{\text{eff}} \) prove to be “almost unitary equivalent.” Hence we can interpret fractions \( \{m/l\} \) in (2) as representing quantized values of two commuting operators \( J_z \) and \( \sqrt{\Delta} \). Let us remark that the approach of Ref. 4 did not use the EKB-quantization of the underlying classical flow explicitly (rather certain “symmetry-reduction” procedures), so it gave no clue as to the semiclassical structure of spectral shifts \( \{ \mu_{lm} \} \) in nonzonal cases.

The present paper aims to uncover such semiclassical structure of \( \{ \mu_{lm} \} \) and to outline an approach to the inverse problem and the isospectrality in the context of integrable or separable potentials \( V \) on \( S^n \). The simplest example is furnished by the quantum Neumann operator:

\[
H = \Delta + V; V = \frac{1}{2} \sum \alpha_i x_i^2 \quad \text{on } S^n.
\]

We remark that quadratic potentials on \( S^n \) share one essential feature of zonal \( \{ V \} \), namely, integrability both on the classical level (due to C. Neumann) and the quantum level (see Sec. II and Ref. 9). The integrability works in the quantum case via explicit integrals \( \{ f_k \} \) of (1), and the EKB-quantization of the classical flow. We pursue this approach in Secs. II–III. The final result (Sec. III) gives semiclassical eigenvalues for the Neumann operator \( H(\hbar) = \hbar^2 \Delta + V \), expressed through a system of hyperelliptic integrals. Thus we get the semiclassical eigenvalues \( \{ \lambda_{im}(\hbar) \} \) for the “small Planck” problem, but such results are not directly applicable to the high energy asymptotics \( \{ \lambda_{im} \} \) as \( l \to \infty \) at \( \hbar = \mathcal{O}(1) \).
Another essential feature of the Neumann problem has to do with its separability (separation of variables), both on the classical level (Hamilton–Jacobi equation) and the quantum Schrödinger problem. The separation exploits a special orthogonal coordinates system on $S^n$, known as ellipsoidal or spheroidal. It is defined by a family of confocal quadrics on $\mathbb{R}^{n+1}$, depending on $(n+1)$ real parameters $\alpha_1<\alpha_2<\cdots<\alpha_{n+1}$. We shall briefly review the ellipsoidal coordinates in Sec. IV and Appendix C. They will be shown to belong to a wider class of separable coordinate systems studied by Stäckel, whose origins go back to Liouville (cf. Ref. 12). A Stäckel system is given by a metric tensor (kinetic energy form) of the type $h(x;p)=\sum g^{ij}(x)p^jp^j$, that allows a complete separation of variables and explicit integration in quadratures. The separation procedure works identically both on the classical level—the geodesic flow of $h$ given by the Hamiltonian–Jacobi equation $h(x;\Psi)=0$, and on the quantum level, i.e., the Laplace–Beltrami operator

$$\Delta = \frac{1}{\sqrt{g}} \sum \partial_i \sqrt{g} \partial_j.$$  

Furthermore, each Stäckel system gives rise to a family of integrable/separable potentials. In the $S^n$-case they take the form

$$V(x_1;\ldots;x_n) = \sum_{i=1}^n \frac{\Pi_j(x_i-\alpha_j)}{\Pi_j'(x_j-x_j)} V_i(x_i);$$  

(3)

with arbitrary one-variable functions $\{V_1(x_1);\ldots;V_n(x_n)\}$, the Neumann Hamiltonian corresponding to a special choice of $\{V_i\}$.

Separation of variables reduces the spectral problem for “Stäckel” Schrödinger operators with potentials (3) to a system of singular ODEs on adjacent intervals $[\alpha_j;\alpha_{j+1}]\mathbb{R}$. An interesting feature of the ellipsoidal reduction is that all ODEs are given by a single differential operator

$$L = \partial^2 + \left[ \frac{1}{2} \sum_{k=1}^{n+1} \frac{1}{x-\alpha_k} \right] \partial + \sum_{k=1}^{n+1} \frac{q_k}{(x-\alpha_k)} \text{a generalized Lamé operator.}$$  

(4)

That contrasts the standard (polar/spherical) separation where different expressions for $\phi, \theta$ appear.

Operator $L$ has $(n+1)$ regular-type singular points at $\{\alpha_j\}$, and a possible irregular singularity at $\{\infty\}$ depending on separation constants $\{q_j\}$. The sphere problem requires all solutions $L[\psi_j]=0$ on intervals $[\alpha_j;\alpha_{j+1}]$ to be regular at both end points. The regularity condition imposes a system of algebraic constraints on separation constants $\{q_j\}$, whose solution could in principle provide a quantized (discrete) set of parameters $\{q_j(m):m=(m_1;\ldots;m_n)\in\mathbb{Z}^n\}$, hence the quantized (exact) spherical eigenvalues,

$$\lambda_m = \sum \alpha_j q_j(m).$$

However, the resulting algebraic system is difficult to write down explicitly let alone to solve in dimensions higher than 1. As for the $S^1$-Neumann problem it is easily converted (via trig. substitution) into the standard Mathieu equation: $\partial^2 - \lambda \cos^2 \theta + E$ on $[0; 2\pi]$. However, the 2D case, $S^2$, poses a more challenging problem. Here (4) is converted into another well-known model the celebrated Lamé–Ince operator

$$\mathcal{L} = \partial^2 - \lambda \sin^2(x;k), \quad \text{or} \quad \mathcal{L} = \partial^2 - \lambda \varphi,$$

with the Weierstrass $g$-function as potential. The $S^2$-Schrödinger problem leads to a perturbation of $\mathcal{L}$, which in the Neumann case $V=eAx \cdot x$ on $S^2$ turns into the so-called Lamé wave equation
\[ L = \partial^2 - \lambda \, \sn^2(x;k) + \epsilon \, \sn^4(x;k). \quad (5) \]

The terminology points to another source of (5)—the reduced wave (Helmholtz) operator \( \Delta + \epsilon^2 \) in ellipsoidal coordinates on \( \mathbb{R}^3 \). The Lamé–Ince operator \( \mathcal{L} \) is well-known in the finite-zone potential theory on \( \mathbb{R}^3 \). Precisely, quantizing the coupling parameter \( \lambda \) to a discrete set of values \( \{ \lambda = l(l+1); \ l = 1; 2; \ldots \} \) yields operators \( \mathcal{L}_l = \partial^2 - l(l+1) \varphi \) having precisely \( l \) zones/gaps in the continuous spectrum on \( L^2(\mathbb{R}) \). The \((2l+1)\) end-points of the gaps \( \{ E_m^l \colon 0 \leq m \leq 2l \} \) correspond to periodic and antiperiodic eigenvalues on \( \mathcal{L} \).

Furthermore, potential \( \lambda \varphi \) and its perturbations \( \lambda \varphi + V \), resulting from the \( S^2 \)-Schrodinger problem turn into double-periodic meromorphic (elliptic) functions on \( \mathbb{C} \). Any such operator, like Lamé’s \( \mathcal{L} \) or its perturbation \( L = \mathcal{L} + V \) can be considered on both the real and the imaginary periods in \( \mathbb{C} \) (Sec. V). So one can look for the double periodic/antiperiodic solutions of the eigenvalue problem. Indeed, the double periodicity condition results from separation on \( S^2 \) and is directly linked to double-periodicity of the Jacobi map \( \Phi : \mathbb{C} \rightarrow S^2 \), that implements the ellipsoidal coordinate change,

\[
\Phi : u + iv \rightarrow \begin{cases} 
x = \sn(u;k) \sn(v;k) \\
y = \cn(u;k) \cn(v;k) \\
z = \dn(u;k) \dn(v;k)
\end{cases}
\quad (6)
\]

Let us elucidate the point by drawing analogy between the \( S^2 \) and the torus case. The torus is obtained by a trigonometric map \( \Phi : \mathbb{R}^2 / \mathbb{Z}^2 \rightarrow \mathbb{T} \times \mathbb{T} \),

\[
\Phi : (x; y) \rightarrow (e^{2\pi i x}; e^{2\pi i y}),
\]

so \( \mathbb{R}^2 \) forms an infinite-sheet cover of \( \mathbb{T}^2 \), the fundamental regions being shifted rectangles \( \{ 0 \leq x, y \leq 1 \} \). Similarly, ellipsoidal coordinates on \( S^2 \) implemented by the Jacobi functions (6), define a \((2K; 2K')\)-double periodic conformal map from \( \mathbb{C} \) modulo period lattice onto the “sphere with a cut.” Once again \( \mathbb{C} \) makes up an infinite-sheet cover of \( S^2 \). The resulting eigenfunctions, \( L[\psi] = E \psi \) extend analytically as periodic/antiperiodic functions in both the real and the imaginary directions.

Our last Sec. V exploits the Lamé eigenvalues and the double-periodicity of the Lamé problem for the asymptotic analysis of spectral shifts \( \{ \mu_{lm} \} \) on \( S^2 \). We observe that the role of the quantized ratios \( \{ ml/l \} \) in the zonal case (2) is played now by the Lamé eigenvalues \( \{ E_{lm}^l \} \).

The appearance of double-periodic problems and the link to the finite-zone theory on \( \mathbb{R} \) suggests a possible approach to nontrivial isospectral deformations on \( S^2 \), by analogy with the well studied torus case.\(^{15}\) Indeed, generic potentials on \( \mathbb{T}^2 \) are well known to be spectrally rigid, but separable-type potentials \( V = V_1(x) + V_2(y) \) clearly allow large (\( \infty \)-D) isospectral deformations, defined by higher KdV-flows of \( V_1 \) and \( V_2 \). Since the ellipsoidal change (6) makes \( S^2 \) to resemble “torus,” one wonders whether similar constructions (deformations) could be implemented on \( S^2 \). We made a preliminary study along these lines, based on the finite-zone potential theory.\(^{13}\) It produced large families of partially isospectral deformations, but also indicated that the finite-zone constraint might be too restrictive to get the complete solution (globally isospectral nontrivial deformations). The complete answer hinges on some unresolved issues in the theory of infinite-zone double periodic potentials “elliptic solitons” (cf., Ref. 16) and would require further study.

The original version of the article appeared in March 1993. Soon afterwards we learned about the recent work\(^9\) by Toth who studied similar problems and proved some of our results (Sec. II). The current revised version completes and clarifies several points left open in the original manuscript, particularly Maslov indices in the semiclassical quantization and the role of the Liouville–Stäckel separation (Sec. III).
II. QUANTIZED NEUMANN PROBLEM

A. Classical conserved integrals

The classical Neumann system defines a Hamiltonian flow on the phase-space $T^*(S^n)$ of the unit sphere $S^n = \{ |x| = 1 \}$ in $\mathbb{R}^{n+1}$ given by Hamiltonian $h(x;p) = p^2 + V(x)$, with quadratic potential

$$V(x) = \sum a_i x_i^2 = A x \cdot x.$$  

Here matrix $A = \text{diag}(\alpha_1; \ldots ; \alpha_n)$ for an arbitrary increasing sequence $0 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n+1}$. So one can view the Neumann Hamiltonian either as a constrained oscillator, or a spherical pendulum in the linear field of force $F = Ax$. Moser gave a concise account on the Neumann problem and showed many interesting connections to the classical Jacobi problem (geodesics on ellipsoids, see also Ref. 1), spectral theory of finite-zone Sturm–Liouville operators on $\mathbb{R}$ and the periodic KdV problem. The Neumann and Jacobi problems are known to be integrable, their commuting integrals defined by polynomial functions

$$f_i(x;p) = x_i^2 + \sum_{j \neq i} \frac{J_{ij}^2}{\alpha_i - \alpha_j} \quad \text{(Neumann)},$$

$$f_i(x;p) = \mu_i^2 + \sum_{j \neq i} \frac{J_{ij}^2}{\alpha_i - \alpha_j} \quad \text{(Jacobi)}.$$  

Here $J = (J_{ij}) = x \times p = (x_i p_j - x_j p_i)$ denotes the angular momentum. The Neumann integrals $\{f_i\}$ are easily verified to obey the relations

$$\sum f_i = x^2; \quad h = \sum \alpha_i f_i = |J|^2 + A x \cdot x.$$  

Those would allow one to constrain the Neumann system from $\mathbb{R}^{2n+2}$ to $T^*(S^n)$, and to express the constraint Hamiltonian as the weighted sum of $n + 1$ commuting integrals (only $n$ of them $\{f_1; \ldots ; f_n\}$ being independent, as $f_{n+1} = 1 - \sum f_i$ on $S^n$). The involutivity/integrability of the constrained system is explained in Appendix A. Our goal here is to quantize the classical Hamiltonian $h$ along with its integrals $\{f_i\}$.

B. Quantization

We want to assign quantum Hamiltonians, operators on $L^2(S^n)$ to classical observables $\{f_i(x;p)\}$ (7) defined on the phase-space $T^*(S^n)$. As above we do it first on the extended phase-space $\mathbb{R}^{2n+2}$. Here the procedure is straightforward

$$f_i(x;p) \rightarrow F_i(x;i\partial) = x_i^2 + \sum_{j \neq i} \frac{J_{ij}^2}{\alpha_i - \alpha_j};$$  

where $J = x \times i \nabla$ represents the angular momentum operator. The $\alpha$-weighted sum of operators $\{F_i\}$ becomes the quantum Hamiltonian

$$H = V(x) + |J|^2; \quad \text{with quadratic potential} \quad V = \sum \alpha_i x_i^2.$$
Next we want to restrict operators \( \{F_i\} \) and \( H \) originally defined on the extended quantum (Hilbert) space \( L^2(\mathbb{R}^{n+1}) \) to functions on the unit sphere \( L^2(\mathbb{S}^n) \), in a way that respects all classical (Poisson) brackets. That could be accomplished via the following general

**Proposition 1:** If vector fields \( \{\xi_1, \ldots, \xi_n\} \) on \( \mathbb{R}^n \) leave a submanifold \( \mathcal{M} \) invariant, then the algebra of differential operators \( \mathcal{A} \), generated by \( \{\xi_i\} \) and all multiplication operators with functions \( \{f(x)\} \) can be restricted on \( \mathcal{M} \), hence on the quantum (Hilbert) space \( L^2(\mathbb{S}^n) \).

It remains only to verify the proper commutation relations of operators \( \{F_i\} \). We start with the simplest 2D case \((n=1)\). Here the operators are \( F_1 = x^2 + (1/\alpha) J^2 \) and \( F_2 = y^2 - (1/\alpha) J^2 \), with \( \alpha = \alpha_1 - \alpha_2 \). Their commutator

\[
[F_1; F_2] = \frac{1}{\alpha} [J^2; x^2 + y^2] = 0.
\]

As for the Hamiltonian

\[
H = \alpha_1 F_1 + \alpha_2 F_2 = J^2 + \alpha_1 x^2 + \alpha_2 y^2 |_{\mathbb{S}} = J^2 + \alpha \cos^2 \theta + \alpha_2,
\]

it turns into the standard Matheau differential operator on \([0; 2\pi]\) with periodic boundary conditions.

Next we consider the 3D case. Introducing parameters \( \alpha = \alpha_1 - \alpha_2; \beta = \alpha_2 - \alpha_3 \); operators \( \{F_i\} \) assume the form

\[
\begin{align*}
F_1 &= x^2 + \frac{J_x^2}{\alpha} + \frac{J_y^2}{\alpha + \beta} \\
F_2 &= y^2 - \frac{J_x^2}{\alpha} + \frac{J_y^2}{\beta} \\
F_3 &= z^2 - \frac{J_z^2}{\alpha + \beta}
\end{align*}
\]

(9)

We apply the standard commutation relations for the \( \text{so}(3) \)-generators (Pauli matrices) \( X; Y; Z \) as well as their squares [elements of the envelope of \( \text{so}(3) \)]:

\[
[X; Y] = Z; [Y; Z] = X; [Z; X] = Y;
\]

\[
[X^2; Y^2] = [Y^2; Z^2] = [Z^2; X^2] = 4XYZ + 2(Y^2 - Z^2 - X^2) = N.
\]

(10)

Now the commutators of operators \( \{F_i\} \) of (9) are easily computed

\[
[F_1; F_2] = \frac{1}{\alpha \beta} [J_2^2; J_1^2] - \frac{1}{\alpha (\alpha + \beta)} [J_2^2; J_1^2] + \frac{1}{\beta (\alpha + \beta)} [J_2^2; J_1^2];
\]

all other terms clearly drop out (commuting operators). Hence, we get by (10)

\[
[F_1; F_2] = \left( \frac{1}{\alpha \beta} - \frac{1}{\alpha (\alpha + \beta)} - \frac{1}{\beta (\alpha + \beta)} \right) N = 0.
\]

**Remark 1:** In the limiting case of coaxial ellipsoids \( (\alpha = \alpha_2) \) one multiplies \( F_1; F_2 \) by \( \alpha = \alpha_2 - \alpha_1 \) and lets \( \alpha \) go to 0. Then \( \alpha F_1, F_2 \to \pm J_\perp^2 \), and one recovers the standard angular momentum \( J_\perp \)-symmetry of such axisymmetric (zonal) potential \( V = \alpha (x^2 + y^2) \).

The above argument is easily extended to higher dimensions and orthogonal algebras \( \text{so}(n) \), so we skip the details.
Our main objective is the spectrum of operator $H$. Existence of the conserved integrals suggests looking for their joint eigenvalues. However, any direct attempt at diagonalizing operators $\{F_i\}$ seems utterly hopeless. So we try next to find a semiclassical approximation of spec $(H)$ based on its classical Hamiltonian flow.

III. SEMICLASSICAL EIGENVALUES AND EBK-QUANTIZATION

The word “quantize” could be used in two different meanings. In the previous section it meant to assign quantum Hamiltonians (operators) $\{F_i\}$ to their classical counterparts $\{f_i\}$. Here the word “quantization” refers to a proper “discretization” of classical integrals to produce eigenvalue spectra.

A. EBK-quantization

We shall follow the general quantization scheme applicable to any classically integrable Hamiltonian of the form $H=h(x;ih\nabla)$ on a cotangent phase-space $T^{*}(M)$ of a Riemannian manifold (configuration space) $M$ (see for instance Refs. 17 and 18). The classical conserved integrals $\{f_j\}$ of the Neumann problem restricted on the unit sphere obey the relation

$$\sum_{i=1}^{n+1} f_i = x^2 = 1. \quad (11)$$

So the phase-space $T^{*}(S^n)$ is foliated into the joint level sets of $n$ independent integrals

$$\Lambda(c) = \Lambda(c_1 \ldots c_n) = \{f_i(x;p) = c_i : 1 \leq i \leq n\}. \quad (12)$$

Furthermore, each joint level set will be shown to consist of a single product-type invariant Lagrangian torus $\Lambda \subset T^{*}(S^n)$ (Appendix C). The latter involves a suitable choice of coordinates (ellipsoidal) and a representation of the Neumann Hamiltonian in the so-called Stäckel form.

To quantize semiclassically Hamiltonians $\{f_j\}$ of (7) by the EBK (generalized Born–Sommerfeld) rules, one picks a basis of fundamental cycles $\{\gamma_j(c) : 1 \leq j \leq n\}$ in each Lagrangian $\Lambda(c)$ and writes down a system of algebraic equations

$$\{ \ldots \}
\begin{equation}
T_j(c) = \oint_{\gamma_j} p \cdot dx = \pi(2m_j + \frac{1}{2} \text{ind}(\gamma_j)).
\end{equation}

Here points $\{m_j\}$ vary over the lattice $Z^n$ and $\text{ind}(\gamma)$ denotes the Maslov (Morse) index of path $\gamma \subset \Lambda(c)$ (see Ref. 17). Solution of system (12) yields a quantized set of parameters $\{c(m) : m = (m_1, \ldots, m_n)\}$, hence a quantized sequence (lattice) of eigenvalues of Hamiltonian $H = f(F_1, \ldots, F_n)$.

$$\lambda_m \approx \hat{f}(c_1(m) \ldots ; c_n(m)) + \ldots ; m \in Z^n. \quad (13)$$

The accuracy of quasiclassical approximation (13) increases with $\hbar \to 0$. To apply the general FKR-scheme, however, one needs to know the fundamental periods $\{T_j(c)\}$ of the action form $p \cdot dx$ on the Lagrangian $\Lambda(c)$ and Maslov indices $\{\text{ind}(\gamma_j)\}$ over fundamental cycles of $\Lambda$. There are no general recipes for doing it, and the answer typically would depend on the nature of the integrable system in question. In our case one can get a closed form representation of periods (12) due to a peculiar feature of the classical Neumann problem, its algebraic integrability to be explained below. Another method involves separation of variables in the Hamilton–Jacobi equation of the Neumann system. It yields both the fundamental periods and their Maslov indices.
B. Ellipsoidal action variables

Following Refs. 2 and 19 we introduce a new set of commuting action-integrals \( \{ \beta_j \} \) to replace \( \{ f_j \} \). These are defined by zeros of the rational function

\[
R(z) = \sum_{j=1}^{n} \frac{f_j}{z - \alpha_j} = \frac{b(z)}{a(z)} = 0 \Rightarrow z = \beta_1; \ldots; \beta_n.
\]

The numerator and denominator of \( R \) are polynomials

\[
a(z) = \prod_{i=1}^{n+1} (z - \alpha_i); \quad b(z) = \prod_{j=1}^{n+1} (z - \beta_j).
\]

New variables \( \{ \beta_j \} \) together with polynomials \( a, b \) give a convenient parametrization of the Neumann problem. Indeed, all relevant quantities, including Hamiltonians \( \{ f_i; h \} \) are simply represented through \( \alpha \)'s and \( \beta \)'s. Namely,

\[
f_k = \text{Res}_{\alpha_k} \left( \frac{b(z)}{a(z)} \right) = \frac{b(\alpha_k)}{a'(\alpha_k)} = \frac{\Pi(\alpha_k - \beta_i)}{\Pi_{i \neq k}(\alpha_k - \alpha_i)},
\]

i.e., \( \{ f_j \} \) are residues of the rational function \( R(z) \). Constraint \( \sum f_i = 1 \) is automatically satisfied by residues \( \{ f_j \} \) of \( R \), by the standard residue Theorem for a pair of polynomials:

\[
a = a_0 z^{n+1} + \cdots \Pi(z - \alpha_i) \quad \text{and} \quad b = b_0 z^n + \cdots
\]

\[
\sum \text{Res}_{\alpha_i} \left( \frac{b}{a} \right) = - \text{Res}_{\alpha_i} \left( \frac{b}{a} \right) = \frac{b_0}{a_0}.
\]

The Hamiltonian is simply expressed through either one of the two families of conserved integrals

\[
h = \sum \alpha_i f_i = \sum_{i=1}^{n} \beta_i \cdot \sum_{i=1}^{n} \alpha_i.
\]

The latter follow once again from the residue calculus applied to rational function \( z(b(z)/a(z)) \). Let us remark that all \( \beta \)'s are real-valued, and in case \( \{ c_i > 0 \} \) they lie in the adjacent intervals separated by \( \{ \alpha_i \} \) (see Fig. 1)

\[
\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_n < \beta_n < \alpha_n.
\]

It turns out\(^2,19\) that each invariant torus \( \Lambda \) of the Neumann problem represents a (real) Jacobian variety of a complex hyperelliptic curve (Riemann surface)

\[
\Gamma = \Gamma(\alpha; \beta) = \{(z, y) : y^2 = P(z) \}
\]

of genus \( n \) defined by polynomial

\[
P(z) = \prod_{i=1}^{n+1} (z - \alpha_i) \prod_{j=1}^{n} (z - \beta_j).
\]

Depending on the mutual location of the \( \alpha \)- and \( \beta \)-zeros of \( P \), determined by signs of \( \{ f_j \} \), we choose a suitable set of \( n \) branch cuts in the complex plane [notice that \( f_{n+1} = 1 - \sum f_j \) is always positive by (II)]. For example, positive coefficients \( \{ c_j \} \) yield all \( \alpha \)'s and \( \beta \)'s intermingled, as in (18), so the cuts are made over intervals: \( \Delta_1 = [\alpha_1; \beta_1]; \ldots; \Delta_n = [\alpha_n; \beta_n] \) (Fig. 1).
The fundamental cycles on Lagrangian $\Lambda$ will be labeled by fundamental cycles on $\Gamma$, chosen as closed loops around branch-cuts $\{\Delta_j\}$. Veselov and Novikov\textsuperscript{19} gave an expression of the action-form $p \cdot dx$ for general classes of “algebraically integrable” systems, i.e., systems whose invariant Lagrangians are parametrized by Jacobians of complex algebraic curves (Riemann surfaces) $\Gamma$ (see Ref. 13). Namely,

$$p \cdot dx = \frac{b(z)dz}{\sqrt{P(z)}} = \frac{\Pi(z-\beta_j)dz}{\sqrt{\Pi(z-\alpha_i)(z-\beta_j)}} = \sqrt{\frac{b(z)}{a(z)}} dz.$$  \hspace{1cm} (20)

The branch-cuts $\{\Delta_j\}$ will be chosen so that polynomial $P$, equivalently fraction $b(z)/a(z)$ inside the square-root (20) remains positive. Hence we get the fundamental periods $\{T_j\}$ as functions of variables $\{\beta_j\},$

$$T_j(\beta) = \int_{\Delta_j} \frac{b}{\sqrt{P}} dz.$$  \hspace{1cm} (21)

The latter are to be quantized by the $EKB$-rules (12) to get a sequence $\{\beta(m)\}$. The resulting sequence of quantized $\beta$'s is then substituted in the classical Hamiltonian $h$ given by (11) to produce semiclassical eigenvalues of the Schrödinger operator $H$,

$$\lambda_m(\alpha) = \sum \alpha_k - \sum \beta_k(m).$$  \hspace{1cm} (22)
C. Invariant Lagrangians, singular projections, and Maslov indices

The complete semiclassical analysis of operators $H(\hbar)$ requires the detail structure of invariant tori $\Lambda \subset T^*(S^n)$ and Maslov indices of fundamental path $\{\gamma_i\}$ on $\Lambda$. In general, the Maslov index of a path $\gamma \subset \Lambda$ depends on the singular set $\Sigma(\Lambda)$ of the projection $Pr: (x, p) \to x$ of $\Lambda$ on the coordinate space $M$ and the intersection of the projected ray $Pr(\gamma)$ with $\Sigma$. We recall that regular points $\{x_0\}$ have the property that projection $Pr$ is one-to-one in the vicinity of $(x_0, p_0)$, so the patch of Lagrangian $\Lambda$ near $(x_0, p_0)$ is locally represented as the graph-surface of a map $x \to p(x)$. In other words, $x \to p(x)$ becomes singular at such $x_0$.

The index of $\gamma$ at an intersection point with $\Sigma$ is equal to the number of coordinates $\{x_i\}$ that change sign on passing through $\Sigma$ along path $\gamma$ and the total index $\text{ind}(\gamma)$ is obtained by summing up indices over all intersection points.

There are no general recipes for computing Maslov indices, and each case requires a specialized treatment. In the case of Neumann Hamiltonian, the analysis is greatly facilitated by introduction of ellipsoidal coordinates on $S^n$ and the St"ackel separation procedure, reviewed in Appendices B and C. Here we shall state the final results.

Ellipsoidal coordinates $\{u_1; \ldots; u_n\}$ vary within a set of adjacent intervals

$$\alpha_1 < u_1 < \alpha_2 < u_2 < \alpha_3 < \cdots < u_n < \alpha_{n+1} \tag{23}$$

obtained by partitioning $R$ with $n+1$ coefficients $\{\alpha_i\}$ of the basic quadratic form (those could always be normalized so that $\alpha_1 = 0$ and $\alpha_{n+1} = 1$). The ellipsoidal coordinates allow a complete separation of variables in the Hamilton-Jacobi equation of the classical Neumann problem, hence its integration in quadratures (see, e.g., Ref. 12). Furthermore, invariant Lagrangian tori $\Lambda \subset T^*(S^n)$ are factored into the product of one-D tori (circles) $\gamma_i$, each one lying in the $i$th coordinate phase-plane in $T^*(S^n)$. These $\{\gamma_i\}$ form a basis of fundamental cycles on $\Lambda$ and their indices are computed in a straightforward manner.

Precisely, the rational function $R(z) = b(z)/a(z)$ of (1), or equivalently the polynomial $P(z) = a(z)b(z)$ of (19) give a convenient parametrization of $\{\gamma_i\}$, namely,

$$\gamma_i: p_i^2 - R(u_i) = 0. \tag{24}$$

Notice, that roots $\{\beta_1; \ldots; \beta_n\}$ of polynomial $b(z)$ and fraction $R(z)$ are conserved integrals of the Neumann flow, that could be used in place of $\{\beta_i\}$ of Sec. 1.

We take a generic torus $\Lambda(\beta)$ and project it down on the configuration space

$$S^n = [\alpha_1; \alpha_2] \times \cdots \times [\alpha_n; \alpha_{n+1}]$$

Since the image must cover an open range of variables $\{u_i\}$ (the condition of nondegeneracy), each $\gamma_i$ of (24) is projected down onto a open subinterval of $[\alpha_i; \alpha_{i+1}]$. The projections determine the admissible values of integrals $\{\beta_i\}$ to ensure the nondegeneracy of $\Lambda(\beta)$. Namely, rational function $R(z) = b/a$ has to assume positive values in each interval $[\alpha_i; \alpha_{i+1}]$.

Hence our task is to distribute $n$ points $\{\beta_i\}$ (roots of polynomial $b$) among $n$ intervals $[\alpha_i; \alpha_{i+1}]$, in such a way that $R$ would take on a positive value in each one of them. A simple combinatorial analysis shows that each $[\alpha_i; \alpha_{i+1}]$ could have no more than two $\beta$'s. All possible configurations are illustrated in Fig. 2. They include:

- Sign-changing branch ($\pm \infty$ at two ends). Here $\gamma_i$ is projected onto a subinterval $[\beta; \alpha]$ below the positive part of the branch.
FIG. 2. Plots of $R(z)$ for different choices of $\beta_1; \beta_2$ (solid dots).

- Up-looking (even) branch without real roots. Here $\gamma_i$ covers the entire interval $[\alpha_i; \alpha_{i+1}]$.
- Down-looking even branch with two real roots $\{\beta_j; \beta_{j+1}\}$, that mark the end-points of the projected $\gamma$.

Let us remark that the up-looking branch with two real roots is excluded, as its presence would force another branch to be strictly negative. Hence there would be no real projection of $\Lambda$ in one of the adjacent coordinate intervals $[\alpha_j; \alpha_{j+1}]$—the degeneracy condition.

Figure 2 illustrates the 2-D case. There are 10 ways to distribute roots $\beta_1; \beta_2$ between 3 $\alpha$'s, hence 10 types of separable tori. The first 6 of those (a–f) are nondegenerate, whereas the remaining 4 are degenerate.

Let us remark that roots $\{\beta_i\}$ determine the singular set $\Sigma(\Lambda)$, made of the product-type "walls" $\Sigma_i : u_i = \beta_i$. Indeed, the Jacobian matrix $\partial p/\partial x$ is diagonal (Appendix C) with entries

$$
\left( \frac{\partial x}{\partial p} \right) = \begin{bmatrix}
\frac{R_1'/\sqrt{R_1}}{} \\
\vdots \\
\frac{R_n'/\sqrt{R_n}}{}
\end{bmatrix}.
$$

Here $R_i$ abbreviates as above $R(u_i)$. So $\Sigma(\Lambda)$ is made either of poles $\{\alpha_i\}$ or zeros $\{\beta_i\}$ of $R$. However, the former $\{\alpha_i\}$ are fictitious singularities, due to parameterization of $S^*$ by variables $\{u_i\}$. Precisely, coordinate intervals $\{\alpha_i < u_i < \alpha_{i+1}\}$ are mapped into great circles $x_i^2 + x_{i+1}^2 = 1$ on
$S^2$, if all but a single $u_j = \alpha_j$ $(j \neq i)$, and they are mapped into spherical ellipsi if one of $u_j \neq \alpha_j$ (see Fig. 3). Depending on the projection range of $\gamma_i$ inside interval $[\alpha_i; \alpha_{i+1}]$ there are three possible cases:

- If $Pr(\gamma_i)$ covers $[\alpha_i; \alpha_{i+1}]$, then no singularities (walls) are encountered in the $i$th direction, i.e., path $\gamma_i$ is projected regularly into two symmetric halves of the “latitudinal” coordinate ellipse in $S^2$ (see Fig. 3 and Fig. 4). Since no singularities are encountered we get $\text{ind}(\gamma) = 0$.

- $\gamma$ is projected onto a subinterval $[\beta_i; \alpha_i]$ or $[\alpha_i; \beta]$. The subinterval parametrizes a symmetric arc of the coordinate spherical ellipse with 2 endpoints corresponding to $\beta$ (Fig. 3). Such $\gamma$ represents a typical Hamiltonian path in a 1-D potential well, each critical point contributes index 1, hence $\text{ind}(\gamma) = 2$.

- $\gamma$ is projected onto a subinterval $[\beta_i; \beta_{i+1}]$ of $[\alpha_i; \alpha_{i+1}]$. Once again $\gamma$ is a typical path in a 1-D potential well, so $\text{ind}(\gamma) = 2$.

This completes the analysis of fundamental path and Maslov indices. The 2-D case (Fig. 2) can be summarized in the Table I.

We shall conclude with the following two comments.

FIG. 4. Ellipsoidal grid on $S^2$ made of two families of confocal ellipsi.
TABLE I. Maslov indices for different types of Lagrangian tori on $S^2$.

<table>
<thead>
<tr>
<th>Types</th>
<th>Fund. cycles</th>
<th>Maslov ind</th>
</tr>
</thead>
<tbody>
<tr>
<td>a,e</td>
<td>[$\alpha \beta$]</td>
<td>[1,1] 2 0</td>
</tr>
<tr>
<td>b,f</td>
<td>[$\alpha \alpha$]</td>
<td>[1,1] 0 2</td>
</tr>
<tr>
<td>c</td>
<td>[$\beta \beta$]</td>
<td>[1,1] 0 2</td>
</tr>
<tr>
<td>d</td>
<td>[$\alpha \beta$]</td>
<td>[1,1] 2 2</td>
</tr>
<tr>
<td>g</td>
<td>[1]</td>
<td>[1] 0</td>
</tr>
<tr>
<td>j</td>
<td>[1,1]</td>
<td>[1] 0</td>
</tr>
<tr>
<td>h,k</td>
<td>[1,1]</td>
<td>[1] 0</td>
</tr>
</tbody>
</table>

1. Formulas (21) express the basic periods $\{T_\beta(\beta)\}$ in terms of the complete hyperelliptic integrals, that generalize the standard elliptic integrals of the first and second kind (see example below). The asymptotic analysis of such integrals, and their dependence on conserved quantities, elliptic moduli $\{\beta_j\}$ could be quite involved, particularly at high energies. But formulas (12) and (22) are fairly straightforward and could be easily implemented numerically to compute semiclassical eigenvalues $\lambda_m(\hbar)$ of $-\hbar^2 A + V$ at low/moderate energy levels.

2. The quantization rules (12) along with the analysis of Maslov indices (Table I) could be extended to degenerate tori (dim $A < n$) along the lines of A. Voros, “The WKB-Maslov method for nonseparable systems,” CNRS Colloq. Int. Geom. Symplect. & Math. Phys. 237, 277–287 (1974). Let us remark that degenerate tori of the Neumann problem cover large (open) regions of the phase-space. In the above description (Fig. 2) they correspond to function $R(z)$ turning strictly negative on certain intervals $[\alpha_i; \alpha_{i+1}]$ [cases (g)-(k)]. Hence they should give a significant input to $\text{spec}(H)$, that would require further study.

Turning to the large eigenvalue problem for operator $H_V = A + V$ the problem could be reduced in the usual way to a semiclassical one $-h^2 A + h^2 V$, where $h = 1/\sqrt{E}$. However, this time small parameter $h$ multiplies potential $V$ as well as $A$, hence the most significant contribution to $\lambda(h)$ comes as a higher order correction to the unperturbed eigenvalue $\lambda(h + 1)$. Such fine cluster structure of spec $(H_V)$ becomes much more involved, and its exact solution would entail a hardly tractable problem of “inversion” and asymptotic analysis of hyperelliptic moduli, the period map of (21). Here we shall illustrate it with the simple 1-D example of $S^1$.

Example: The quantum Neumann Hamiltonian on $S^1$ is the classical Mathieu operator on $[0; 2\pi]$ with periodic boundary conditions.

$$H = -\partial^2 + (\alpha_2 - \alpha_1)\sin^2 \theta + \alpha_1.$$ Fixing a high energy level $\lambda$ and writing the standard Bohr–Sommerfeld condition for the action-integral with potential $V = \alpha \sin^2 \theta$, $\alpha = \alpha_2 - \alpha_1$, we get

$$T = \int_0^{2\pi} \sqrt{(\lambda - \alpha_1) - \alpha} \sin^2 \theta \, d\theta = \sqrt{(\lambda - \alpha_1)} \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta.$$

Here small parameter $k = \sqrt{\alpha/\lambda}$ becomes the modulus of the standard elliptic integral of the 2nd kind $E(k)$. Clearly, function $T(\lambda)$ admits an expansion in powers of $k^2 = \alpha/\lambda$.

$$T = \sqrt{(\lambda - \alpha_1)} \left\{ \frac{\pi}{2} - \frac{\alpha}{2\lambda} \int \sin^2 \theta + \cdots \right\}. \tag{25}$$ The latter can be compared to formula (21) that gives $T$ in terms of the Novikov–Veselov integral $\int \sqrt{b(z)/a(z)}$. Taking large negative $\beta < \alpha_1$ and writing
we get another standard elliptic integral form for $T(\beta)$,

$$
\sqrt{\alpha_2 - \beta} \{ \lambda (sn^{-1}(1;k)) - cn(sn^{-1}(1;k)) \}, \text{ of modulus } k = \frac{\alpha_2 - \alpha_1}{\alpha_2 - \beta}.
$$

(26)

Remembering the relation between $\lambda$ and $\beta = -\lambda + (\alpha_1 + \alpha_2)$, and comparing modulus $k$ of (25) with the one of (26) we see that the $EKB$ (Bohr–Sommerfeld) rule is consistent with the Veselov–Novikov formula (21).

That concludes our discussion of semiclassical quantization procedures based on the classical Hamiltonian flow. In the next section we shall adopt an approach based on separation of variables. Separation will reduce the multi-D spectral problem on $S^n$ to certain ODE problems like the Matheau and the generalized Lamè equations. The connection between the classical and quantum problems will play an essential role here as well. We shall exploit it to rederive the fundamental periods and Maslov indices used in semiclassical formulas (12) above.

IV. ELLIPSOIDAL COORDINATES AND STÄCKEL–ROBERTSON SEPARATION

A. Ellipsoidal coordinates

Ellipsoidal coordinates in $\mathbb{R}^{n+1}$ are defined by a family of confocal quadrics

$$
R(z) = \sum_{i=1}^{n+1} \frac{x_j^2}{z - \alpha_j} = 1.
$$

(27)

A polynomial equation (27) has $n + 1$ roots

$$
\{ z_i = x_i(x_1; \ldots; x_{n+1}) : i = 1; \ldots; n + 1 \}
$$

that give ellipsoidal coordinates of point $x \in \mathbb{R}^{n+1}$. Replacing constant 1 by 0 in the r.h.s. of (27) we get the family of asymptotic cones to the quadrics

$$
R(z) = \sum_{i=1}^{n+1} \frac{x_j^2}{z - \alpha_j} = \frac{b(z)}{a(z)} = 0.
$$

(28)

These cones carve up a coordinate grid on $S^n$, called ellipsoidal or spherico-conal (Fig. 4). So ellipsoidal coordinates $\{u_1; \ldots; u_n\}$ are zeros of rotational function $R(z)$ with fixed poles $\{\alpha_i\}$ in the denominator $a(z) = \Pi_{i=1}^{n+1}(z - \alpha_j)$, whose numerator $b(z) = b_0 \Pi_{i=1}^{n+1}(z - u_j)$ has principal coefficient $b_0 = \Sigma x_i^2 = 1$ on the unit sphere. Coordinates $\{u_i\}$ parametrize sphere $S^n$ and give convenient expression to other relevant quantities in terms of polynomials $a; b$. Namely,

1. Cartesian coordinates are residues of $R(z)$ at $\{\alpha_i\}$:

$$
\cdots x_i^2(u_1; \ldots; u_n) = \text{Res}_{\alpha_i} \left[ \frac{b(z)}{a(z)} \right] = \frac{b(\alpha_i)}{a'(\alpha_i)} = \frac{\Pi(\alpha_k - u_i)}{\Pi(\alpha_k - \alpha_i)}.
$$

(29)

2. The "unit sphere" constraint $\{\Sigma x_i^2 = 1\}$ is automatically satisfied by residues $\{x_i^2\}$, provided the leading coefficient $b_0 = 1$;

3. quadratic (Neumann) potential is simply expressed through the new variables
As in the previous section both relations are easily derived by the Calculus of residues applied to fractions $b(z)/a(z)$ and $z[b(z)/a(z)]$, respectively. Variables $\{u_i\}$ can be shown to form an orthogonal coordinate system on $S^n$, each $u_i$ varying over interval $[\alpha_i; \alpha_{i+1}]$ (Appendix B). As for the Neumann potential $V$ it could be represented in two different ways. One of them is obtained through the expansion in residues of $zb(z)/a(z)$

$$V = - \sum_{i=1}^{n} u_i + \sum_{i=1}^{n+1} \alpha_i \frac{b(\alpha_i)}{a'(\alpha_i)}.$$  \hspace{1cm} (31)

On the other hand taking residues of fraction $p(z)/b(z)$ with numerator

$$p(z) = z^n - \left(\sum_{i=1}^{n} \alpha_i\right)z^{n-1} + \cdots$$

we get

$$V = - \sum_{i=1}^{n} \frac{p(u_i)}{b'(u_i)}.$$  \hspace{1cm} (32)

We remark that the lower order terms of $p$ do not contribute to $V$, as they sum to 0 by the residue Theorem. Representations (30)–(32) of $V$ are crucial for the separation procedure described below both for the (classical) Hamilton–Jacobi equation and the quantum Neumann Hamiltonian.

Let us illustrate formulas (30)–(32) in the two-sphere case, parametrized by pairs of variables $\alpha_1 \leq u \leq \alpha_2 \leq v \leq \alpha_3$ (see Figs. 4 and 5)

$$\begin{align*}
    x &= \sqrt{\frac{(\alpha_1 - u)(\alpha_1 - v)}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}} \\
    y &= \sqrt{\frac{(\alpha_2 - u)(\alpha_2 - v)}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)}} \\
    z &= \sqrt{\frac{(\alpha_3 - u)(\alpha_3 - v)}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}}
\end{align*}$$  \hspace{1cm} (33)
The Neumann potential on $S^2$ becomes

$$V(u,v) = -(u + v) + \sum_{i=1}^{3} \alpha_i = -\frac{u^2 - (\Sigma \alpha_i)u}{u - v} - \frac{v^2 - (\Sigma \alpha_i)v}{v - u}. \quad (34)$$

**B. Laplacian in ellipsoidal coordinates**

The ellipsoidal Laplacian has diagonal form due to orthogonality (Appendix B), the corresponding metric-tensor being $ds^2 = \Sigma g_{ii} du_i^2$. So

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \partial_i \sqrt{g} \partial_i g_{ii}.$$ 

The diagonal coefficients are

$$g_{ii}(u) = \frac{b'(u_i)}{a(u_i)} > 0,$$

and $g = \prod g_{ii}$ denotes the standard determinant of matrix $(g_{ij})$. In the ellipsoidal case on $S^n$ the determinant becomes

$$g = \frac{W_n^2}{a_1 \cdots a_n}. \quad (35)$$

Here and henceforth notation $a_i; b'_i$ will be used to abbreviate the values of polynomials $a(z)$ and $b'(z)$ at points $z = u_i$

$$a_i = a(u_i) = \prod_{k \neq i} (u_i - \alpha_k),$$

$$b'_i = b'(u_i) = \prod_{k \neq i} (u_i - u_k).$$

The numerator of (35) contains square of the standard Vandermond determinant

$$W_n = W(u_i, \ldots u_n) = \prod_{i<j} (u_i - u_j)$$

that comes from the product of derivatives $\{b'_i\}$. The Laplacian in our notations takes the form

$$\Delta = \frac{\sqrt{a_1 \cdots a_n}}{W} \sum_{i=1}^{n} \partial_i \left\{ - \frac{W}{\sqrt{a_1 \cdots a_n}} \frac{a_i}{b'_i} \right\} \partial_i.$$ 

A specific example of $S^2$ yields

$$\Delta = \frac{\sqrt{-a(u_1)a(u_2)}}{u_2 - u_1} \left\{ \partial_1 \left( \sqrt{-\frac{a(u_1)}{a(u_2)}} \partial_1 + \partial_2 \left( \sqrt{\frac{a(u_2)}{a(u_1)}} \partial_2 \right) \right) \right\}.$$ 

Let us remark that the products (fractions) inside the square roots are all positive due to opposite signs of $a(u)$ at points $u_1; u_2$. The ellipsoidal metric tensor and the corresponding Laplacian belong to a wider class of separable metric Hamiltonians, studied by Stäckel (see Refs. 10 and 11).
C. Stäckel–Robertson separation for Laplacians and Schrödinger operators

Stäckel studied metric-type (kinetic energy) Hamiltonians \( h = \frac{1}{2} \Sigma \ g^{ij}(x)p_ip_j \) that allow a complete separation of variables for the Hamilton–Jacobi (HJ) equation

\[
 h(x; \partial S) = E
\]

hence explicit integration (in quadratures) of the corresponding Hamilton flow. The details of the Hamilton–Jacobi method and the Liouville and Stäckel separation procedures are reviewed in Appendix C. Here we shall briefly state the main results. The class of Stäckel Hamiltonians is determined by the so-called Stäckel matrix

\[
 \sigma = \begin{bmatrix}
 \sigma_{11}(x_1) & \cdots & \sigma_{1n}(x_1) \\
 \sigma_{21}(x_2) & \cdots & \sigma_{2n}(x_2) \\
 \vdots & \ddots & \vdots \\
 \sigma_{n1}(x_n) & \cdots & \sigma_{nn}(x_n)
 \end{bmatrix};
\]

each row of \( \sigma \) depending on a single variable \( (x_i) \) —for the \( i \)th first row. The Stäckel metric tensor \( ds^2 = \frac{1}{2} \Sigma \ g_{ij}dx_i^2 \) and the corresponding (kinetic energy) Hamiltonian \( h = \frac{1}{2} \Sigma \ g^{ii}p_i^2 \) \( (g^{ii} = 1/g_{ii}) \) are diagonal with coefficients \( \{g^{ii}\} \) determined by the first row entries of the inverse Stäckel matrix \( \sigma^{-1} = [\sigma^{ij}] \), i.e., \( g^{ii} = \sigma^{-1} \). Stäckel Hamiltonian can be perturbed by any potential of the form

\[
 V = \sum V_i \sigma^{1i},
\]

with arbitrary one-variable functions \( \{V_i(x_i)\} \). So the general form of a separable Stäckel Hamiltonian is

\[
 h = \frac{1}{2} \Sigma \ \sigma^{1i}(p_i^2 + V_i).
\]

The separation of the classical HJ equation proceeds as follows. One writes

\[
 \sum_i \ \sigma^{1i}[(\partial_i S)^2 + V_i] = E
\]

and observes that summation \( (38) \) gives the first row of \( \sigma^{-1} \) multiplied by the column-vector \( \{p_i^2 + V_i\} \). Supplementing \( (38) \) with the remaining rows of \( \sigma^{-1} \) one can write the entire product as

\[
 \sigma^{-1} \begin{pmatrix}
 p_1^2 + V_1 \\
 \vdots \\
 p_n^2 + V_n
 \end{pmatrix} = \begin{pmatrix}
 c_1 \\
 \vdots \\
 c_n
 \end{pmatrix},
\]

where the r.h.s. is made of separation constants (conserved integrals) \( \{c_j\} \), the first one being energy \( c_1 = E \). Multiplying both sides by \( \sigma \) one ends up with a sequence of separated ODE equations

\[
 (\partial_i S)^2 + V_i = \sum_j c_j \sigma_{ij}(x_i)
\]

for the \( i \)th variable \( x_i \). Hence follows the explicit (quadrature) solution of the HJ equation in Stäckel's coordinates, Eq. (75) of Appendix C.
Robertson extended the Stäckel's separation method to the corresponding quantum objects, Laplacians and Schrödinger operators

\[ \Delta = \frac{1}{\sqrt{g}} \sum_i \partial_i \sqrt{g} g^{ii} \partial_i \quad \text{and} \quad H = -\Delta + V \]

with separable potentials \( V \). He assumed the Stäckel form of the metric-tensor, i.e., diagonal entries

\[ g^{ii} = \frac{\sigma^i}{\sigma} \]

for a suitable matrix \( \sigma \), plus an extra condition on the product of coefficients. Namely, the inverse determinant

\[ \frac{1}{g} = \prod_i g^{ii} = \frac{\prod_i \sigma^{ii}}{\sigma^{n-2}} = \prod_i f_i(x_i) \]

factors into the product of one-variable functions \( \{f_i\} \). Equivalently, the log-derivative of \( g^{ii} \sqrt{g} \) depends on \( x_i \) only,

\[ \frac{\partial}{\partial x_i} \log(g^{ii} \sqrt{g}) = f_i(x_i) \tag{40} \]

It turns out that the corresponding Laplacian (Schrödinger) eigenvalue-problem

\[ H[\psi] = \lambda \psi \]

also allows a separation of variables, i.e., eigenfunction \( \psi \) factors into the product of one variable functions \( \psi = \prod \psi_i(x_i) \), that satisfy certain ODEs. The separation proceeds similar to the classical HJ case. One writes

\[ \frac{1}{\psi_i} H[\psi_i] = \sum_{i=1}^n g^{ii} \frac{1}{\psi_i} \left[ \partial_i^2 + f_i \partial_i + V_i \right][\psi_i] = \lambda, \]

with coefficients \( \{f_i\} \) of (40), and observes that the l.h.s. is equal to the first entry of the matrix product \( \sigma^{-1} \) by the column-vector made of the ODE-expressions \((1/\psi_i) H_i[\psi_i] \). As above we augment it by the remaining rows and rewrite the product via inverse Stäckel matrix

\[ \sigma^{-1} \begin{bmatrix} \frac{1}{\psi_i} H_i[\psi_i] \\ \vdots \\ \frac{1}{\psi_i} H_i[\psi_i] \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \tag{41} \]

As above the r.h.s. is made of separation constant \( \{c_i\} \) the first of which gives the eigenvalue (energy) of the quantum Hamiltonian \( c_1 = \lambda \). Multiplying both sides of (41) with Stäckel \( \sigma \) we end up with an ODE in variable \( u_i \) for each factor \( \psi_i \) of

\[ H_i[\psi_i] = \left( \partial_i^2 + f_i \partial_i + V_i + \sum_j c_j \sigma_{ij}(x_i) \right)[\psi_i] = 0. \tag{42} \]
Suitable boundary conditions should be introduced for (42), depending on the geometry of the original problem. Those in turn will impose some algebraic constraints on separation constants \( \{c_i\} \), sitting in the potential part of each \( H_i \). Solution to the resulting algebraic system should give a suitable set of value to \( \{c_i\} \), in particular, the eigenvalues \( c_i = \lambda \) of \( H \).

\section{The Neumann Hamiltonian on \( S^n \)}

We shall now implement the Stäckel-Robertson separation in ellipsoidal coordinates on the sphere \( S^n \). The metric coefficients in ellipsoidal variables \( \{u_i\} \) were shown to be

\[ g^{ij} = \frac{a_i}{b_j^2}; \quad \text{with} \quad g = \det(g_{ij}) = \frac{W_n^2}{\prod a_i}, \quad (43) \]

Here we abbreviated \( a_i = a(u_i), \quad b'_i = b'(u_i), \) as in Sec. IV B, and remembered that \( \prod b'_i = W_n^2 \) — the square of the Vandermond determinant. Hence follows the \( i \)th diagonal entry of the metric tensor

\[ g^{ii} \sqrt{g} = \sqrt{\frac{a_i}{\prod b_j^2}}. \]

The primed product \( \prod' \) indicates that the \( i \)th factor is dropped. Now coefficient \( f_i \) of the reduced ODE (43) becomes

\[ f_i = \frac{1}{2} \frac{a_i'}{a_i} = \partial_i \log a = \frac{1}{2} \sum_{i=1}^{n+1} \frac{1}{u_i - \alpha_k}. \]

One could directly verify that the Stäckel matrix in ellipsoidal coordinates on \( S^n \) is given by

\[ \sigma = \begin{bmatrix} \frac{u_1^{n-1}}{a_1} & \frac{u_1^{n-2}}{a_1} & \cdots & 1 \\ \frac{u_2^{n-1}}{a_2} & \frac{u_2^{n-2}}{a_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{u_n^{n-1}}{a_n} & \frac{u_n^{n-2}}{a_n} & \cdots & 1 \end{bmatrix}; \quad a_i = a(u_i). \]

Next we recall the Neumann potential expanded in \( \alpha \)'s and \( u \)'s (31)

\[ V = \sum_i^{n+1} \alpha_i - \sum_i^n u_i = - \sum_i \frac{p(u_i)}{b_i'(u_i)}; \]

with a polynomial \( p(u) = u^n + (\Sigma \alpha_j) u^{n-1} \) in the numerator. We observe that the \( i \)th term of the sum can be written as \( g^{ii} V_i \) in the Stäckel convention (43). Hence, follows (remembering that \( g^{ii} = a_i'/b_i^2 \))

\[ V_i = \frac{p_i}{a_i} = \frac{p(u_i)}{a(u_i)}. \]

Combining \( V_i \) with the separation potential derived from the \( i \)th row of matrix \( \sigma \) via (3.15) we get
Polynomial $C(u)$ of degree $n$ in the numerator has the leading coefficients $c_0 = \epsilon$-coupling constant of $H = -\Delta + \epsilon V$, while all other $\{c_j : 1 \leq j \leq n\}$ are Stäckel–Robertson separation constants. The first of them

$$c_1 = \lambda - \sum_{i=1}^{n+1} \alpha_i$$

gives the Schrödinger eigenvalue. Now we can write down the reduced Neumann ODE,

$$H_i = \partial_i^2 + \left[ \frac{1}{2} \sum_{k=1}^{n+1} \frac{1}{u_i - \alpha_k} \right] \partial_i + \frac{C(u_i)}{a(u_i)}. \tag{44}$$

Rational function $C(u)/a(u)$ can be expanded in simple fractions (poles $\{\alpha_k\}$),

$$\frac{C(u_i)}{a(u_i)} = \sum_{k=1}^{n+1} \frac{q_k}{u_i - \alpha_k},$$

with residues

$$q_k = \frac{C(\alpha_k)}{b'(\alpha_k)}.$$

The sum of residues $\sum_{k=1}^{n+1} q_k = 1$, or $\epsilon$, depending on whether $V$ is taken as $\Sigma \alpha_k x_i^2$ or $\epsilon \Sigma \cdots$. So there are $n$ independent variables $\{q_i\}$, uniquely determined by coefficients $\{c_j\}$ of $C$. The Neumann spectral problem is thus reduced to a single ordinary differential operator depending on $2n+1$ parameters $\{\alpha_1 < \alpha_2 < \cdots < \alpha_{n+1}\} \text{ and } \{q_1 ; \cdots ; q_n ; q_{n+1}\}$, subject to $\Sigma q_i = 1$ (or $\epsilon$),

$$H = \partial^2 + \left[ \frac{1}{2} \sum_{k=1}^{n+1} \frac{1}{u - \alpha_k} \right] \partial + \sum_{k=1}^{n+1} \frac{q_k}{u - \alpha_k} \tag{45}$$

considered on different intervals $[\alpha_i ; \alpha_{i+1}]$. So each operator $H_i$ of (44) coincides with $H$ restricted on the $i$th interval with proper boundary conditions at the end-points. Those boundary conditions together with the basic algebraic constraint $\Sigma q_i = 1$ should give a system of $(n+1)$ algebraic equations for unknown separation parameters $\{q_i\}$. Once $\{q_i\}$ are found the eigenvalue $\lambda$ of the original $S^n$-problem is computed by

$$\lambda = \sum_{i=1}^{n+1} \left( \alpha_i - q_i \sum_{j \neq i} \alpha_j \right) = \sum_{i=1}^{n+1} \alpha_i q_i. \tag{46}$$

Remark: Formula (46) closely resembles an expansion of the Neumann Hamiltonian $h$ in terms of the conserved integrals $\{f_j\}$ (Sec. II) $h = \Sigma \alpha f_j$. In this respect $\{q_j\}$ behave like the "joint eigenvalues" of the commuting quantum operators $\{F_j\}$. It is unclear whether such a coincidence reveals some hidden connections between PDE operators $\{F_j\}$ on the sphere (Sec. II) and the reduced ODE-operators $\{H_i\}$, as there is no apparent direct relation between both.

Stäckel separable potentials on $S^n$.

Each differential operator $H_i$ can be augmented by a potential function $V_i(u_i)$ that comes from a separable Stäckel Hamiltonian (37) of the form
For $n = 2$ we get

$$V = \frac{a(u_1)}{u_1 - u_2} V_1(u_1) + \frac{a(u_2)}{u_2 - u_1} V_2(u_2);$$

where $a(u) = (u - \alpha_1)(u - \alpha_2)(u - \alpha_3)$. As we already mentioned the quadratic Neumann potential is a particular example corresponding to identical functions $\{V_i\}$

$$V_i(u) = \cdots = V_n(u) = \frac{p(u)}{a(u)} \quad \text{where} \quad p = u^n + \left( \sum \alpha_j \right) u^{n-1}.$$
The entries of the transfer matrix depend on parameters \( \{q\} \) only, the off-diagonal entry being

\[
B = B_i(q) = \left\{ \frac{\psi_2; \psi_1}{\phi_1; \psi_1} \right\}
\]

here \( \{ \cdots; \cdots \} \) denotes the Wronskian of two solutions. To get a regular solution at both end-points (i.e., \( \psi_2 = A \psi_1 \)) coefficient \( B \) must be 0. That gives an algebraic relation for parameters \( \{q_j\} \), hence a system of \( n \) algebraic equations.

\[
\begin{align*}
\cdots & B_i(q) = B(\alpha_i; \alpha_{i+1}; q) = 0. \\
\cdots & 
\end{align*}
\] (48)

Those being supplemented with the basic constraint \( \sum q_i = 1 \) (or \( \epsilon \)) the resulting algebraic system should in principle yield a discrete (quantized) set of separation constants \( \{q_j(m)\} \), hence the quantized eigenvalues \( \{\lambda_m\} \) of the Neumann Hamiltonian. However, Eqs. (48) are hard to write down in any closed form, let alone solve. So next one tries to bring ODE (45) to some conventional form amenable to such analysis. The first natural choice is the Hill operators on \( R \) with periodic potentials.

2. The Mathieu–Hill problem

Equation (45) with two singular points \( (n = 1) \) could be converted to the standard Mathieu equation via a trigonometric substitution \( z = (\alpha_2 - \alpha_1) \cos 2x \). That is the form of the 1-D Neumann problem. It turns out that higher-D ordinary differential operators \( H \) could also be converted to the Hill’s form by a combination of the conjugation and the change of variables. Given a differential operator \( H = (1/p) \partial \rho \partial + \cdots \) conjugation with \( \sqrt{\rho} \) brings \( H \) into a symmetric (self-adjoint) form

\[
\sqrt{\rho} H \frac{1}{\sqrt{\rho}} = \partial^2 - \left( \frac{1}{2} \frac{\log \rho}{\rho}' \right)^2 + \frac{1}{2} \frac{\log \rho}{\rho}''.
\]

In our case (45) function \( \rho \) has the product-form

\[
\rho = \prod (z - \alpha_j)^{1/2},
\]

made of different factors \( \rho_i = \sqrt{|z - \alpha_i|} \). Each factor \( \rho_i \) will eliminate the \( i \)th partial fraction \( 1/(z - \alpha_i) \) in the first order coefficient of \( H \) and leave the rest of them intact. Making a particular choice,

\[
\rho = \prod_{i \neq 1, \alpha_j} \rho_j = \prod'' \rho_j;
\]

operator \( H \) is brought into the form

\[
\partial^2 + \frac{1}{2} \left( \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_1} \right) \partial + \frac{3}{16} \sum'' \frac{1}{(z - \alpha_j)^2} + \sum \frac{q_j - x_j}{z - \alpha_j},
\] (49)

with coefficients.
\[ \chi_i = \frac{1}{16} \sum_{j \neq i} \frac{1}{\alpha_i - \alpha_j}. \]

Here the double-primed products and sums \( \Pi''; \Sigma'' \) indicate the omitted pair of factors (or summands) \( \{(z - \alpha_i); (z - \alpha_{i+1})\} \).

The trigonometric substitution \( z = b \cos^2 x \) with \( b = b_i = \alpha_{i+1} - \alpha_i \) (length of the \( i \)th interval), transforms (49) into the Hill’s operator \( \delta^2 + W \), with a periodic potential on \([0; \pi]\),

\[ W(x; q) = \delta^2 + W_0 + \sum_{1}^{\infty} W_m \cos(2mx). \]

Fourier coefficients \( \{W_m\} \) are linear combinations of separation constants \( \{q_j\} \), that could be computed explicitly in terms of geometric parameters \( \{\alpha_j\} \).

Turning to the boundary conditions we notice that regularity at the singular end-points \( \{\alpha_i; \alpha_{i+1}\} \) for operator \( H_i \) (49) corresponds to the choice of even Hill’s eigenfunctions \( \psi = \sum_{0}^{n} A_m \cos(2mx) \), made of even Fourier modes \( \{\cos(2mx)\} \), while singular solutions are transformed into odd ones: \( \psi = \sum_{0}^{n} B_m \cos((2m + 1)x) \) (see Ref. 21). In the simplest 1-D case \( H \) is reduced to the standard Mathieu operator

\[ H = \delta^2 + 2b(q_1 + q_2) \cos 2x + 2b(q_2 - q_1), \quad \text{with} \quad b = \alpha_2 - \alpha_1, \]

i.e., Hill’s potential \( V = A \cos(2x) + B \).

After operators \( \{H_i\} \) are converted to the Hill’s form, we proceed to rewrite the original spectral problem in terms of Hill’s eigendata for each \( H_i \)

\[ H_i = \delta^2 + V^i(q) + U^i(x; q); \quad \text{where} \quad U^i = \sum_{1}^{\infty} W_m(q) \cos(2mx). \]

For the sake of convenience the zero Fourier mode \( V = W_0 \) is separated from the rest of the series and the remaining terms are combined into a single function \( U(x; q) \). All Fourier-coefficients \( \{V^i; U^i_m\} \) are linear combinations of separation constants \( \{q_j\} \) that could be computed in the closed form, but the expressions become fairly cumbersome, so we will not bring them here. Now we can choose an \( n \)-tuple of even eigenvalues \( E_{2m_1}(U^1); E_{2m_2}(U^2); \ldots; E_{2m_n}(U^n) \) for each of operators \( H_1; \ldots; H_n \), and write a system of algebraic equations for \( \{q_j\} \)

\[ \begin{cases} V^1(q) = E_{2m_1}(U^1) \\ V^2(q) = E_{2m_2}(U^2) \\ \vdots \\ V^n(q) = E_{2m_n}(U^n) \\ \sum_{1}^{n+1} q_j = 1 \text{(or } \epsilon) \end{cases} \]

Solution of system (50) could in principle yield a quantized set of separation constants \( \{q_j(m); m = (m_1; \ldots; m_n)\} \), hence the quantized eigenvalues of the Schrödinger problem with an arbitrary separable potential \( V \) on \( S^d \) in the form

\[ \lambda_m = \sum_{1}^{n} \alpha_i q_i(m). \]

Let us conclude this section with two comments.
1. System (50) could be compared to (48) or to the semiclassical equations (22) of Sec. III. Unlike the Neumann case (22) it applies to arbitrary separable potential. It is more explicit than the former two, the l.h.s. being made of certain linear functions of \( q_i \)'s, while the r.h.s. involves the set of “even eigenvalues” of the given Hill potentials, also depending linearly on \( \{ q_i \} \). Eigenvalues of the Hill’s problem \( \{ E_m(u) \} \) have well-known asymptotic expansion to any order \(^{21-23}\)

\[
E_m(u) = m^2 + U_0 + \frac{1}{m^4} \int (U - U_0)^2 + \cdots .
\]

However, the data \( \{ q_i \} \) enters \( \{ E_m \} \) in such a way that higher order corrections depend on higher powers of \( U \), hence higher powers of “large parameters \( \{ q_i \} \).” It is not clear whether system (50) could be truncated (approximated) by a polynomial one, and then how one should deal with the resulting polynomial (nonlinear) equations to extract a meaningful asymptotic information on the structure of \( \{ \lambda_m \} \).

2. The apparent drawbacks of the Hill approximation (50) could be linked to the underlying geometric deficiency. Indeed, the 1-D Mathieu–Hill operator represents the exact \( S^1 \)-Neumann problem. So system (50) could be viewed as coming from some sort of “circular (1-D)” approximations of the \( n \)-sphere Hamiltonian \( \Delta + V \), based on adjacent pairs of parameters \( \{ \alpha_i; \alpha_{i+1} \} \). The higher-D Laplacians however, may not be well represented or approximated by the 1-D “circular” objects. So the next natural step would be to consider adjacent triplets of parameters \( \{ \alpha_i-1; \alpha_i; \alpha_{i+1} \} \) and the resulting Lamè operators. Geometrically that would correspond to “approximating” higher-D spherical Laplacians by the \( S^2 \)-ones. Indeed, the Lamè equation itself comes from the \( S^2 \)-Laplacian, the separable Schrödinger problem being reduced to the perturbed Lamè equation. In the next section we shall carry out such analysis in the 2-D case. The “triplet” Lamè representation shows clear advantages over the Hill’s representation (50). It reveals in particular the cluster structure of spec \( (\Delta + V) \), and yields asymptotics of individual spectral shifts \( \{ \mu_{mn} \} \) in terms of potential \( V \) and the Lamè eigendata. It also suggests a possible approach to iso-spectral deformations on \( S^2 \) via the finite and infinite-zone potential theory on \( R \).

V. LAMÈ EQUATION AND THE \( S^2 \) SPECTRAL PROBLEM

A. Lamè eigenmodes

From now we shall focus on the 2-sphere case and use the Jacobi form of ellipsoidal coordinates on \( S^2 \). The latter are defined by three real parameters \( \alpha_1 < \alpha_2 < \alpha_3 \) and relative distances: \( a = \alpha_2 - \alpha_1 \) and \( b = \alpha_3 - \alpha_2 \). The coefficients \( \{ a; b \} \) determine elliptic moduli

\[
k = \sqrt{\frac{a}{a+b}} ; \quad \text{and its dual} \quad k' = \sqrt{\frac{b}{a+b}}.
\]

The elliptic change of variables,

\[
z = \text{sn}^2(x;k) \quad \text{on} \quad [0; a] \quad \text{and} \quad z = \text{sn}^2(y;k') \quad \text{on} \quad [0; b],
\]

takes an algebraic Lamè equation

\[
H = \partial^2 + \frac{1}{2} \left[ \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_2} + \frac{1}{z - \alpha_3} \right] \partial - \frac{\varepsilon z^2 + (\lambda - \varepsilon \alpha)z + c}{4(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)} + \cdots \quad (51)
\]

into a pair of perturbed Lamè operators

\[
L = \partial^2 + E - \lambda \text{ sn}^2 + V(\text{sn}^2),
\]

FIG. 6. The real and imaginary periods of the double-periodic Lamè problem.

\[ L' = \partial_\gamma^2 + E - \lambda \ \text{sn}^2 + V'(\text{sn}'^2). \]

Henceforth we shall abbreviate our notations for the cnoidal Jacobi functions

\[ \text{sn} = k \ \text{sn}'(x; k); \quad \text{sn}' = k' \ \text{sn}'(y; k'). \]

The resulting potential \( \lambda \ \text{sn}^2 + V \) is double-periodic with real and imaginary half-periods given by the complete elliptic integrals of the 1st kind

\[ K = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \quad \text{and} \quad K' = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k'^2 \sin^2 t}}. \]

One can show that the regular boundary condition of the algebraic Lamè operator (51) corresponds to \( 2K, 2K' \)-periodic or antiperiodic conditions for two equations

\[ L[\psi] = 0; \quad L'[\psi'] = 0. \]

Double periodicity will be shown to impose strong algebraic constraints on spectral parameter \( E \) and coupling constant \( \lambda \) of (52). We shall start with the unperturbed case, i.e., Laplacian \( \Delta \) on \( S^2 \) in ellipsoidal coordinates. There (52) turns into a pair of unperturbed Lamè operators

\[ \mathcal{L}_\lambda = \partial_\gamma^2 + E - \lambda \ \text{sn}^2, \]

\[ \mathcal{L}' = \partial_\gamma^2 + E - \lambda \ \text{sn}'^2, \]

in fact a single Lamè operator considered on two periods (see Ref. 24): the real one \([0; 2K] \), and the imaginary one \([K; K+2iK'] \) (Fig. 6). Both operators share an identical set of parameters: the coupling (separation) constant \( E \) and the spherical eigenvalue \( \lambda \).

The double-periodic solutions of the Lamè equation are known to exist only for a discrete set of the coupling constant \( \{\lambda = l(l+1): l = 1, 2, \ldots\} \). For any such \( l \) spectral parameter \( E \) takes on \( 2l+1 \) values \( \{E_m(k): m = 0; 1; \ldots; 2l+1\} \), half of which \( l \) or \( l+1 \), depending on parity of \( l \) corresponds to periodic (cnoidal) eigenfunctions \( \{\text{Ec}_m\} \) of the Lamè operator \( \mathcal{L}_\lambda \) (53) while the
other half gives antiperiodic \( sn \)-oidal ones \( \{Es_m^q\} \) (Fig. 7). The eigenvalues could also be labeled \( \{E_m^+\} \) (even) and \( \{E_m^-\} \) (odd), depending on the parity of the corresponding eigenfunctions with even and odd eigenvalues intermingled.

Here we adopt the Ince's notations for the Lamè functions, those should be read ellipsoidal-cn or ellipsoidal-sn, by analogy with the Jacobi elliptic functions. Each Lamè function \( \{Ec_m^n;Es_m^q\} \) can be written as a product of \( sn^n cn^m dn^m \) (with exponentials \( \alpha, \beta, \gamma = 0; 1 \)) times a Lamè polynomial \( P(sn^2) \) of degree \( \frac{1}{2}(l - \alpha - \beta - \gamma) \). Furthermore, the \( m \)th eigenfunction of operator \( \mathcal{S} \) on the real period (53) is analytically extended through the \( (l - m) \)-th eigenfunction of \( \mathcal{S}' \) on the imaginary period. Since index \( m \) counts the number of oscillations (zeros) of an eigenfunction (for any Sturm–Liouville problem) we see that the total number of the real and imaginary zeros remain constant for any Lamè function \( \{Ec_m^n;Es_m^n\} \). The corresponding eigenfunctions of the Laplacian, called ellipsoidal (or spherico-conal) harmonics are made of products of each species with complementary quantum numbers \( m \) and \( l - m \),

\[
\psi_l^m(x,y) = \begin{cases} 
Ec_l^m(x)Ec_l^{l-m}(y); m = 0; 1; \ldots; l; & x \in [0; 2K] \\
or \\
Es_l^m(x)Es_l^{l-m}(y); m = 1; 2; \ldots; l; & y \in [K; K + 2K']
\end{cases}
\] (54)

The complementary indices \( m \) and \( l - m \) that appear in (54) could be interpreted in terms of the oscillation properties of spherical eigenmodes. Namely, the total number of the “horizontal” and “vertical” oscillations in the \( x \); \( y \) variables (total angular momentum) remains constant \( l \). Here one could draw a close parallel between ellipsoidal harmonics \( \{\psi_l^m\} \) and the standard spherical harmonics \( \{Y_l^m(\phi; \theta)\} \). The latter are factored in the product of the Fourier mode in \( \theta \) and an associated Legendre function (polynomial) in \( \phi \): \( Y_l^m = e^{im\phi}P_l^m(\cos \phi) \), the role of two Lamè operators being played by the angular momentum \( i\partial_{\theta} \) and the associated Legendre operator \( \partial_{\phi}^2 + \cot \phi \partial_{\phi} - m^2/\sin^2 \phi \). Here the same complementarity relation holds as the number of latitudinal oscillations complements the longitudinal one.
Remark: To dispel a possible confusion about the “finitude” of the Lamè spectrum \( \{E_m\} \) let us make the following comment. Each of two operators: the real \( \mathfrak{D} \) and the imaginary \( \mathfrak{D}' \) have infinite periodic/antiperiodic spectra on \( \mathbb{R} \). The double-periodicity requirement, however, imposes rigid constraints on the joint set of energy-coupling parameters \( \{E, \lambda\} \) due to the analytic continuation property of \( \{E_m, E'_m\} \). It is hardly surprising then to find out that the “coupled problem” possesses only a finite (possibly void) set of solutions.

B. Perturbed eigenvalues

The Lamè–Ince operator \( \mathfrak{D} \) gives the well known example of a finite \( l \)-zone potential \( V = l(l+1)\varphi \) on \( \mathbb{R} \), its periodic/antiperiodic eigenvalues \( \{E_m\} \) marking the ends of stability/instability intervals (zones/gaps) of the continuous spectrum. Spectra of such operators can be analyzed within the framework of the \( l \)-zone potential theory (see Ref. 13). Ellipsoidal separation of Sec. IV brings the \( S^2 \)-Laplacian \( \Delta \), the Neumann operator \( -\Delta + \Sigma \alpha_i x_i^2 \), as well as more general separable Hamiltonians \( \{H_v\} \),

\[
V = \frac{a(u)}{u-v} V(u) + \frac{a(v)}{v-u} V'(v); \quad a(z) = (z - \alpha_1)(z - \alpha_2)(z - \alpha_3);
\]

to the perturbed Lamè problem. The eigenvalues and eigenfunctions of the \( l \)th cluster of \( H_v \), \( \lambda_l = \{\lambda = l(l+1) + \mu_{lm}\} \), would correspond to periodic/antiperiodic solutions of the resulting pair:

\[
L = L_\lambda = \delta^2 - \lambda \text{ sn}\,^2 + V \quad \text{on real period} \quad [0; 2K],
\]

\[
L' = L'_\lambda = \delta^2 - \lambda \text{ sn}'^2 + V' \quad \text{on imaginary period} \quad [K; K+2iK'].
\]

To abbreviate further our notations we shall call unperturbed Lamè potentials

\[
q_0 = l(l+1)sn^2; \quad \text{and} \quad q'_0 = l(l+1)sn'^2;
\]

while their perturbations

\[
\delta q = q - q_0 = \mu q_0 + V; \quad \text{and} \quad \delta q' = q' - q'_0 + \mu q_0' + V'.
\]

The perturbed eigenvalues of \( L \) and \( L' \) will be denoted by

\[
\{E_m = E_m(\delta q): 0 \leq m \leq 2l\} \quad \text{and} \quad \{E'_m = E'_m(\delta q'): 0 \leq m \leq 2l\}.
\]

The cluster number is fixed here, so we drop a superscript \( l \) in \( E_m \). The unperturbed Lamè–Ince eigenvalues \( \{E_i(0)\} \) and \( \{E'_i(0)\} \) obey the relation

\[
E_m(0) = E'_{-m}(0); m = 0, 1, \ldots, l.
\]  \hspace{1cm} (55)

Precisely, one breaks \( 2l+1 \) eigenvalues \( \{E_m\} \) into the periodic and antiperiodic halves and let \( m \) vary in the range: \( 1 \leq m \leq l \), or \( 0 \leq m \leq l \). So (55) pairs eigenvalues of \( E \) and \( E' \) of the like parity: periodic-to-periodic, antiperiodic-to-antiperiodic. Perturbed eigenfunctions retain the basic properties of the Lamè functions, namely,

\[
L[\psi_m] = E_m \psi_m; \quad L'[\psi'_{-m}] = E'_m \psi'_{-m}.
\]  \hspace{1cm} (56)

Though \( \psi \) and \( \psi' \) need not analytically extend each other, unless perturbation potentials are given by the same function \( V = V' \), two sets of spectral parameters: eigenvalues \( \{E\} \) and coupling constant \( \lambda = l(l+1) + \mu \) (i.e., separation constant and spherical eigenvalue) are identical in both
equations (56). The latter allows one to write an equation for spectral shifts \( \{ \mu = \mu_{lm} \} \) in terms of the perturbed eigenvalues \( \{ E_m \}; E_{i-m} \} \), namely,

\[
E_m(\mu q_0 + V) = E_{i-m}(\mu q_0 + V').
\]

(57)

For any fixed \( l \) and \( m \) (57) gives the \( m \)th shift in the \( l \)th cluster \( \mu_{lm} \). Formula (57) could be compared to the results of Secs. III–IV. It also reduces the multi-D spectral problem to certain ODE (Sturm–Liouville) operators. The main advantage of (57) however is that each spectral shift \( \mu_{lm} \) is produced from a single algebraic equation (57) rather than a coupled nonlinear system.

Equations (57) could be expanded in terms of small coupling parameter \( \epsilon \), i.e., perturbation \( H_\epsilon = -\Delta + \epsilon W \) with a separable potential

\[
W = \frac{a(u)}{u-v} V(u) + \frac{a(v)}{v-u} V'(v)
\]

(58)
on \( S^2 \) to produce shifts \( \{ \mu_{lm}(\epsilon) \} \) to any degree of accuracy in powers of \( \epsilon \). Here we shall do it in the linear approximation

\[
\mu_{lm} \sim \left\{ \frac{\delta E_{i-m}}{\delta q} V' - \left( \frac{\delta E_m}{\delta q} V \right) \right\} \left/ \left\{ \frac{\delta E_{i-m}}{\delta q} q_0 - \left( \frac{\delta E_m}{\delta q} q_0 \right) \right\} \right.
\]

(59)

As above \( E_m, E_{i-m} \) denote the \( m \)th and \( (l-m) \)-th eigenvalues of \( L \) and \( L' \), respectively, and all variational derivatives \( \delta E/\delta q; \delta E'/\delta q \) are evaluated at \( \epsilon = 0 \), in other words the unperturbed “Lamè points” \( q_0 = \lambda^m \) and \( q'_0 = \lambda \) \( sn'^m \).

Formula (59) could be further specified by recalling that any eigenvalue problem \( L_\psi = E \psi \) has derivative \( \delta E/\delta q \) equal to the square of the corresponding (unperturbed) eigenfunction \( |\psi|^2 \). In our case \( \psi \) coincides with the \( m \)th Lamè function, \( \psi^m \), \( \psi^m \sim Ec^m(z;k) \) (periodic) or \( Ec^m(z;k) \) (antiperiodic).

So for any (\( \epsilon \))-small perturbation we get

\[
\mu_{lm}(\epsilon) = \epsilon \frac{\langle |\psi^m|^2 |V' \rangle - \langle |\psi^m|^2 |V' \rangle' - \langle |\psi^m|^2 |V' \rangle' - \langle |\psi^m|^2 |V' \rangle' - \langle |\psi^m|^2 |V' \rangle' - \langle |\psi^m|^2 |V' \rangle'}{\langle |\psi^m|^2 |sn'^2 \rangle - \langle |\psi^m|^2 |sn'^2 \rangle'} + \mathcal{O}(\epsilon^2).
\]

(60)

Here \( \langle \cdots \rangle \) and \( \langle \cdots \rangle' \) refer to the \( L^2 \)-inner products on the real and complex periods \( [0, 2K] \); \( [0, 2K'] \). Formula (60) gives linearized spectral shifts \( \{ \mu_{lm} \} \) of \( H_\epsilon \) explicitly represented in terms of the Lamè eigenfunctions \( \{ \psi^m \} \) and perturbation parameters \( \{ V; V' \} \).

As a corollary we obtained a “linearized” version of local spectral rigidity (cf. Ref. 4) for large classes of separable potentials on \( S^2 \), like polynomial and other pairs \( \{ V = P(sn^2); V' = P'(sn'^2) \} \). Precisely,

\textbf{Proposition 2: Given any separable potential \( W \) on \( S^2 \) (58) with polynomial functions \( \{ V = P(u); V' = P'(v) \} \) the linearized spectral shifts \( \{ \mu_{ml} \} \) (60) of Schrödinger operator \( H = -\Delta + \epsilon W \) uniquely determine \( W \).

Indeed, the latter depends on finitely many parameters (coefficients), that enter both functions \( \{ V; V' \} \) linearly. Hence (60) turns into a system of linear equations for unknown coefficients for \( P \) and \( P' \) that could possess a unique solution.

\textbf{Remark:} Formula (60) suggests that the unperturbed Lamè eigenvalues \( \{ E_m = E'_m \} \) should play the role of rationals \( \{ m/l \} \) in the zonal case of Ref. 4, so one should be able to represent \( \mu_{lm} \sim A(E_m) + \cdots \) with suitable function coefficients \( \{ A(x); \cdots \} \). The explicit form of such \( A \)
would require further analysis. As for the asymptotics of \( \{ E_m^\prime \} \) some partial results are known for small \( E \ll \lambda \). Those could be derived by the standard WKB analysis of the large-coupling problem, \( H = \partial^2 + \lambda q \), \( \lambda \gg 1 \). Indeed, the \( m \)th eigenvalue of such operator can be expanded as
\[
[E_m \sim m \sqrt{\lambda} q_0^\prime + \cdots]
\]
where \( q'' \) is taken at a critical (minimum) point of the potential well \( q \). So variational derivative \( \partial E_m/\partial q \) is asymptotic to \( \text{Const} \). Hence at the left/right ends of the range \( \{ 0 \leq m \leq l \} \) the Lamé eigenvalues \( \{ E_m^\prime \} \) behave like fractions, \( m^2 / l(l+1) = ml/\lambda \). Of course, the latter applies only to low eigenvalues \( E_m^\prime \) (resp. \( m \ll \sqrt{\lambda} \)), whereas the Lamé spectrum \( \{ E_m : 0 \leq m \leq 2l \} \) covers the entire range \( \{ 0 ; l(l+1)/2 \} \) and is distributed in a nonuniform fashion.

We plan to return to asymptotic analysis of (60) and the nature of function \( A(E) \) leading approximation of \( \{ \mu_{lm} \} \) elsewhere. Here let us just mention that a related asymptotic problem for the Lamé wave equation:
\[
\frac{\partial^2}{\partial t^2} + E - l(l+1) \sin^2 + \omega^2 \sin^4
\]
was studied in Ref. 3.

C. Comments

- An interesting issue suggested by separable potentials on \( S^2 \) has to do with possible non-trivial iso-spectral deformations by analogy with the torus case \( T^2 \). We made some preliminary study based on the finite-zone theory of Ref. 13 and got partial results. These partial results however demonstrated limitations of the finite-zone theory, and showed that the complete solution would involve certain “infinite-zone limits” of finite-zone potentials. Such issues have not yet been addressed in the infinite-zone theory to our knowledge.

- Though large iso-spectral classes on \( S^2 \) are conceivable, the sphere problem clearly exhibits more rigidity, than the torus one. Indeed, separable potentials on \( S^2 \) that allow iso-spectral deformation (if they exist) would be quite exceptional among all (generic) functions \( W(u;v) \). As for generic \( S^n \) potentials we believe them to be spectrally rigid.

- Finally, let us briefly mention the higher-D Neumann-type spectral problems on \( S^n(n \geq 3) \), particularly semiclassical asymptotic of \( \{ \mu_{km} \} \). Here the reduced (separated) problem becomes a generalized Lamé equation with four (or more) singularities. Unlike the classical Lamé case very little is known about its solutions. One possible approach would be to develop the analysis based on the “three-point Lamé approximations” of a generalized Lamé equation.

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VI. APPENDICES

A. Integrability of the Neumann system

We shall show that the constrained Hamiltonian \( H = V(x) + |x \times p|^2 \) with quadratic \( V = \sum \alpha_i x_i^2 \) on the unit sphere coincides with the Neumann’s \( \mathcal{H} \), and integrals \( \{ f_i \} \) form a Poisson-commuting family. Our exposition will follow the elegant method of Ref. 2. Given a diagonal matrix \( A = \text{diag}(\alpha_1, \ldots, \alpha_n) \) we consider a family of quadratic forms \( Q_z \) on \( \mathbb{R}^n \) depending on complex parameter \( z \),
\[
Q_z(x) = (z - A)^{-1} x \cdot x,
\]
and define a family of Hamiltonians on the phase-space $\mathbb{R}^{2n}$

$$\Phi_z = \frac{1}{2} Q_z(x) + \frac{1}{2} \{Q_z(x) Q_z(p) - Q_z(x; p)\},$$

where $Q_z(x; p)$ is the corresponding bilinear form. One verifies directly that family $\{\Phi_z\}$ is commuting

$$\{\Phi_z; \Phi_w\} = 0, \text{ for all } z; w \text{ in } \mathbb{C}.$$

For any fixed point $(x; p)$ in $\mathbb{R}^{2n}$ function $\Phi_z$ becomes meromorphic (rational) in variable $z$ with poles at $\{\alpha_i\}$. Its residues are precisely integrals $\{f_i\}$

$$\Phi_z = \frac{1}{2} \sum_{i=1}^{n} f_i \frac{1}{z - \alpha_i}.$$

This explains involutivity of $\{f_i\}$ on $\mathbb{R}^{2n}$, hence integrability of $h = \Sigma \alpha_i f_i$. Next we want to constrain system $\{f_i\}$ and Hamiltonian $h$ on the unit sphere and its phase-space

$$T^*(S^n) = \{(x; p): x^2 = 1; x \cdot p = 0\}.$$

We observe that geometric constraint $f = (x^2 - 1) = \Sigma f_i$ is itself a commuting integral, while its canonical partner $g = p \cdot \nabla f = x \cdot p$ has a nonvanishing bracket with $f$,

$$\{f; g\} = |\nabla f|^2 = 1 \text{ on } S^n.$$

In such a setup the reduction procedure of Ref. 2 gives a new family of commuting constrained Hamiltonians

$$h^* = h - \lambda f; \quad f_i^* = f_i - \mu_i f,$$

with multipliers

$$\lambda = \frac{\{h; g\}}{\{f; g\}} = \frac{Ax \cdot x}{x^2}; \quad \mu_i = \frac{\{f_i; g\}}{\{f; g\}}.$$

The particular form of multipliers is immaterial here, as long as $\{h; f; f_i\}$ commute. Geometrically, (61) provides corrections to the original Hamiltonians in such a way that the restricted flows of $h$ and $\{f_i\}$ on the constrained manifold $\{f = 0; g = 0\}$ extend to commuting flows in the ambient space $\mathbb{R}^{2n+2}$.

**B. Orthogonality of ellipsoidal coordinates**

We want to compute gradients of the ellipsoidal coordinate functions $x_i(u)$ of (29). Those could be expressed through rational functions

$$b_i(z) = \frac{b(z)}{(z - u_i)^2} = \frac{1}{z - u_i} \prod_{j \neq i} (z - u_j)$$

as

$$\frac{\partial x_k}{\partial u_i} = \sqrt{\frac{b_i(\alpha_k)}{(\alpha_k - u_i) a'(\alpha_k)}}, \quad i = 1; \ldots; n.$$

Now the orthogonality relations for gradients vectors $X_i = (\cdots \partial x_i/\partial u_i, \cdots)$ take the form
\[ X_i^2 = \sum_{j=1}^{n} \frac{b_j(\alpha_j)}{a'(\alpha_j)} = -\frac{b'(u_i)}{a(u_i)}, \tag{62} \]

\[ X_i X_j = \sum_{i=1}^{n} \frac{1}{a'(\alpha_k)} = 0. \tag{63} \]

The former (62) results from the residue count of \( b_i(z)/a(z) \), the latter (63) comes from residues of \( b_{ij}(z)/a(z) \), whose numerator

\[ b_{ij}(z) = \frac{b(z)}{(z-u_i)^2(z-u_j)^2} = \frac{1}{(z-u_i)(z-u_j) \prod_{k \neq i; j} (z-u_k)}. \]

It remains to observe that the \( u_i \)- and \( u_j \)-residues cancel each other in \( b_{ij}(z)/a(z) \), whereas the sum of all \( \alpha \)-residues gives the requisite dot-product.

### C. Hamilton–Jacobi equation and Stäckel separation

#### 1. General procedure

We shall briefly review the separation procedure for the Hamilton–Jacobi (HJ) equation defined by a classical Hamiltonian \( h(x;p) \) on a 2n-phase space \( \mathcal{P} \). The action function \( S \) of \( h \) solves a HJ differential equation

\[ h(x;\partial S) = E \text{ - const} \tag{64} \]

subject to proper boundary conditions. One looks for a family of solutions depending on parameters \( \{c_1; \ldots; c_n\} \) —the constants of integration, the maximal admissible number being \( n \). The resulting \( n \)-parameter solution \( S(x;c) \) becomes a generating function of the canonical transform on \( \mathcal{P} \)

\[ \Phi: (x,p) \rightarrow (y;c), \tag{65} \]

where the new canonical variables \((y;c)\) are obtained by solving the system

\[
\begin{align*}
p &= S_x(x;c), \\
y &= -S_y(x;c).
\end{align*}
\]

The first equation (66) could be solved for \( c = c(x;p) \) provided \( \det(S_{xx}) \neq 0 \). The resulting map (65) conjugates \( h \) to a new Hamiltonian, depending on action variables \( \{c_i\} \) only

\[ h \circ \Phi = E(c). \]

In some cases the Hamiltonian \( h \) itself could be chosen as one of them, e.g., \( h = E = c_1 \). Now the Hamiltonian flow of \( h \) is explicitly resolved in terms of the conserved action variables \( \{c_i\} \) and the canonically conjugate angle variables \( \{y_i\} \). Indeed, algebraic system

\[
\begin{align*}
y_i(x;c) &= -\frac{\partial S}{\partial c_i} = y^0_i - \text{const} \\
t - t_0 &= -\frac{\partial S}{\partial E} (x;c)
\end{align*}
\]

yields the flow line \( x = x(t;E,c,y^0) \) at a fixed energy \( E \) and conserved integrals \( \{c_i\} \) as a function of the initial data \( \{y^0_i\} \).
2. The Liouville case

Liouville (see Ref. 12) studied Hamiltonian flows that one could integrate explicitly in quadratures. He looked at the standard “kinetic+potential” Hamiltonians and came up with class \{h\} defined by two families of functions \{\sigma_i(x_i)\} and \{V_i(x_i)\}, each one depending on a single variable,

\[
h(x;p) = \frac{1}{2} \sum_i \frac{1}{\sigma} (p_i^2 + V_i),
\]

where \(\sigma = \sum \sigma_i\). The corresponding HJ equation

\[
\sum \frac{(S_i^2 + V_i)}{\sigma} = E
\]

clearly allows a separation of variables for the action function \(S = \sum S_i(x_i; c_i)\). Here each \(S_i\) solves an ODE

\[
(S_i')^2 + V_i - E\sigma_i = c_i, \quad (67)
\]

whence

\[
S_i = \int \sqrt{c_i + E\sigma_i - V_i} \, dx_i.
\]

Let us also notice that the conserved integrals \(\{c_i\}\) are subject to the constraint \(\sum c_i = 0\) due to (67). So the Liouville Hamiltonian allows an explicit quadrature solution

\[
\begin{align*}
\frac{\partial S_i}{\partial c_i} &= \int_{x_i^0}^{x_i} \frac{dx_i}{\sqrt{f_i}}, \\
\cdots \\
\frac{\partial S}{\partial E} &= \sum_i \int_0^{x_i} \frac{\sigma_i dx_i}{\sqrt{f_i}} = t - t_0
\end{align*}
\]

(68)

with functions \(f_i = c_i + E\sigma_i - V_i\) inside the radicals.

3. Stäckel Hamiltonians

Stäckel (see Ref. 10) extended the Liouville’s theory to a larger class of “diagonal” metric Hamiltonians \(h = \frac{1}{2} \sum g^{ii} p_i^2\), based on the notion of Stäckel matrix

\[
\sigma = \begin{bmatrix}
\sigma_{11}(x_1) & \cdots & \sigma_{1n}(x_1) \\
\sigma_{21}(x_2) & \cdots & \sigma_{2n}(x_2) \\
\cdots & \cdots & \cdots \\
\sigma_{n1}(x_n) & \cdots & \sigma_{nn}(x_n)
\end{bmatrix}
\]

(69)

Entries in each row of \(\sigma\) depend on a single variable: \(x_1\)—for the first row, \(x_2\)—for the second one, etc. The coefficients \(\{g^{ii}\}\) of \(h\) are given by the first row of the inverse Stäckel matrix \(\sigma^{-1}\)

\[
g^{ij} = \sigma^{ij},
\]
where $\sigma^{ij} = M_{ji}(\sigma)/\det(\sigma)$, in terms of minors (cofactors) of $\sigma$. The reciprocal coefficients $g_{ii} = 1/g^{ii}$ define the corresponding Stäckel metric tensor

$$ds^2 = \frac{1}{2} \sum g_{ii} dx_i^2.$$ 

Stäckel metric Hamiltonian (kinetic energy) can be perturbed with any potential of the type

$$V = \sum V_i \sigma^{1i},$$

where $\{V_i(x_i)\}$ are arbitrary single-variable functions as in the Liouville’s case. So a general separable Stäckel Hamiltonian has the form

$$h = \frac{1}{2} \sum_{i=1}^{n} \sigma^{1i}(p_i^2 + V_i).$$

The separation of the Stäckel’s HJ equation proceeds similarly to the Liouville’s case. We write

$$\sum_i \sigma^{1i}[(\partial_i S)^2 + V_i] = E$$

and observe that summation $\Sigma_i \cdots$ could be viewed as the product of first row of the inverse Stäckel matrix by the column vector $v = \{p_i^2 + V_i\}$. Supplanting (72) with the remaining rows of $\sigma^{-1}$ we can write the product as

$$qv = (\sigma^{-1} v) = (c_1, \ldots, c_n).$$

The separation constants $\{c_i\}$ in the r.h.s. can be chosen arbitrarily, the first of them being energy $c_1 = E$. Multiplying both sides by $\sigma$ we get a sequence of separated equations

$$(\partial_i S)^2 + V_i = \sum_j c_j \sigma_{ij}(x_i)$$

an ODE in the $i$th variable $x_i$. Hence follows the separation of the HJ equation in Stäckel’s coordinates,

$$S_i(x_i) = \int f_i dx_i; \quad \text{with} \quad f_i = \sum_j c_j \sigma_{ij}(x_i) - V_i$$

and explicit (quadrature) solution of the flow (68)

$$\begin{cases}
\frac{\partial S_i}{\partial c_i} = \int_{x_i}^{x_i} \sigma_{ii} dx_i \\
\vdots \\
\frac{\partial S}{\partial E} = \sum_i \int \frac{\sigma_{ii} dx_i}{\sqrt{f_i}} = t - t_0
\end{cases}$$

Furthermore, functions

\[ h_i = \sum_j \sigma^{-1}(p_j^2 + V_j) \]  

(summation over the \(i\)th column of \(\sigma\))

are easily verified to form a maximal Poisson-commuting family for Stäckel Hamiltonians

\[ h = \Sigma s g^{ii} p_i^2 \text{ and } \Sigma g^{ii} (p_i^2 + V_i). \]

In particular, the Stäckel metric form \(h\) gives rise to an integrable geodesic flow.

4. Stäckel form of the Liouville and Neumann problems

We shall show that the Liouville and Neumann Hamiltonians belong to the Stäckel class, and exhibit Stäckel matrices for both. Let us remark that the Stäckel class exhausts all possible separable metric-form Hamiltonians in \(n\) variables under certain natural conditions (see Ref. 10).

In the Liouville’s case the Stäckel matrix has \(\{\sigma_i\}\) in the first column, ones on the upper subdiagonal and \{-1\} in the last row,

\[
\sigma = \begin{bmatrix}
\sigma_1 & 1 \\
\sigma_2 & \ddots \\
\vdots & & 1 \\
\sigma_n & -1 & \ldots & -1
\end{bmatrix}
\]

Here integrands \(\{f_i\}\) of (74) are given by \(f_i = \sigma_i E + c_i\) so the conserved vector in the r.h.s. of (73) is made of \((E; c_1; \ldots; c_{n-1})\).

In the Neumann case entries of \(\sigma\) depend on polynomial \(a(z) = \Pi(z - \alpha_j)\). We shall abbreviate the values of \(a\) at points \(\{u_1; \ldots; u_n\}\) by \(a_i = a(u_i)\). Then the Stäckel matrix becomes

\[
\sigma = \begin{bmatrix}
u_{1}^{n-1} & u_{1}^{n-2} & \ldots & 1 \\
a_1 & a_1 & \ddots & \vdots \\
u_{2}^{n-1} & u_{2}^{n-2} & \ldots & a_2 \\
a_2 & a_2 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
u_{n}^{n-1} & u_{n}^{n-2} & \ldots & a_n \\
a_n & a_n & \ddots & a_n
\end{bmatrix}
\]

The corresponding conserved vector (73) is \((c_1 = E; c_2; \ldots; c_n)\) and the integrands assume the fractional form

\[
f_i = \sum_{j}^{n-1} \frac{c_{n-j} u_j}{a_j} + \frac{c u_i^{n}}{a_i}
\]

the first term representing the metric (kinetic) component of the Hamiltonian (74), the second resulting from the Neumann potential. The latter represents the value of rational function

\[ R(z) = \frac{b(z)}{a(z)} \]

at \(z = u_i\) used throughout the paper, particularly Secs. III–IV.
5. Integration and classical periods

We remark that Stäckel separation yields explicit momentum variables \( p = p(x;c) \) on the joint level set of \( \{c_i\} \), Lagrangian \( \Lambda(c) \), the fundamental path \( \{\gamma_i\} \) on \( \Lambda \) and periods of the action form \( p \cdot dx \) along \( \{\gamma_i\} \), needed in the semiclassical quantization of Sec. III. Indeed, by (66)

\[
p_i = \frac{\partial S}{\partial x_i} = \pm \sqrt{f_i}.
\]

So the fundamental paths are coordinate tori

\[
\gamma_i : p_i^2 = f_i \quad \text{in the } (x_i;p_i)-\text{phase plane}
\]

and the corresponding classical periods

\[
T_i(c) = \int_{\gamma_i} p \cdot dx = 2 \int \sqrt{f_i} \, dx_i.
\]

In the Neumann case \( f_i = R(u_i) \), so we rederive the Veselov–Novikov formula for \( \{T_i\} \) as

\[
\int \sqrt{b/a} \, dz \quad \text{over the positive branch cut of rational function } R(z) - b/a.
\]

---

1. Another closely related classical problem, the geodesic flow on ellipsoid: \( A^{-1} x \cdot x = 1 \), studied by Jacobi, is also integrable. In fact both problems (Neumann and Jacobi) could be transformed one into the other: H. Knörrer, J. fur die Reine und Ang. Math. 334, 69–78 (1982).


