1. (a) Find the general solution of \( \frac{dy}{dt} = 2ty \).

This is a separable equation, so we separate variables:

\[
\int \frac{1}{y} \, dy = \int 2t \, dt
\]

\[
\ln |y| = t^2 + C
\]

\[
|y| = e^{t^2+C} = e^C e^{t^2}
\]

\[
y = \pm e^C e^{t^2}
\]

Since \( y = 0 \) is also a solution,

\[
y = k e^{t^2}
\] is the general solution.
(b) Verify that \( y(t) = (\sin t)e^{t^2} \) is a solution of
\[
\frac{dy}{dt} = 2ty + (\cos t)e^{t^2}.
\]

Just use the Product Rule:

\[
dt \left[ (\sin t)e^{t^2} \right] = (\cos t) e^{t^2} + (\sin t) e^{t^2},
\]
and
\[
2ty + (\cos t)e^{t^2} = 2t(\sin t)e^{t^2} + \cos t)e^{t^2},
\]
so
\[
\frac{dy}{dt} = 2ty + (\cos t)e^{t^2}.
\]

(c) Find the solution of the initial value problem
\[
\frac{dy}{dt} = 2ty + (\cos t)e^{t^2}, \quad y(0) = 4.
\]

This is a linear equation.

In part (a) we found the general solution of the homogeneous equation \( \frac{dy}{dt} = 2ty \): \( y_h = ke^{t^2} \).

In (b) we saw that \( y_p = (\sin t)e^{t^2} \) is a particular solution of the original equation.

So the general solution is \( y = (\sin t)e^{t^2} + ke^{t^2} \).

By the initial condition, \( y = 0 + k \cdot 1 \), so the solution to the IVP is \( y = (\sin t + 4)e^{t^2} \).
2. Consider the modified logistic model

\[
\frac{dP}{dt} = P \left(1 - \frac{P}{4} \right) \left(\frac{P}{2} - 1 \right).
\]

(a) Find all the equilibrium solutions of this differential equation.

The RHS is 0 when \( P=0, 4, 2 \), so these are the equilibria.

(b) Draw the phase line for this equation, and identify the equilibria as sinks, sources, or nodes.

\[
\begin{align*}
5 \left(1 - \frac{5}{4}\right) \left(\frac{5}{4} - 1\right) &< 0 \\
3 \left(1 - \frac{3}{4}\right) \left(\frac{3}{4} - 1\right) &> 0 \\
1 \left(1 - \frac{1}{4}\right) \left(\frac{1}{4} - 0\right) &< 0 \\
-1 \left(1 + \frac{1}{4}\right) \left(-\frac{1}{4} - 0\right) &> 0
\end{align*}
\]

0 and 4 are sinks. 2 is a source.
(c) What will be the long-term behavior of the population in each of the following situations?

i. $P(0) = 1$

The population will decrease and approach 0.

ii. $P(0) = 3$

The population will increase and approach 4.

iii. $P(0) = 5$

The population will decrease and approach 4.

(d) What is the significance, for a population described by this model, of the population size 2?

A population larger than 2 (even only a little larger) will survive, whereas a population smaller than 2 (even only a little smaller) will not. So 2 is a threshold value, where the long-term fate of the population changes drastically.
3. (a) Verify that \( y_1(t) = \frac{1}{\sqrt{2t+1}} \) and \( y_2(t) = \frac{1}{\sqrt{2t+4}} \) are both solutions of

\[
\frac{dy}{dt} = -y^3.
\]

Use the Laplace Principle:

\[
\frac{dy_1}{dt} = \frac{d}{dt} (2t+1)^{-1/2} = -\frac{1}{2} (2t+1)^{-3/2}. 2
\]

\[
= -\frac{1}{(2t+1)^{3/2}} = -y_1^3
\]

and \( \frac{dy_2}{dt} = \frac{d}{dt} (2t+4)^{-1/2} = -\frac{1}{2} (2t+4)^{-3/2}. 2 \)

\[
= -\frac{1}{(2t+4)^{3/2}} = -y_2^3.
\]

(b) *Without* finding the general solution of the differential equation, what can you say about solutions of \( \frac{dy}{dt} = -y^3 \) for which the initial condition \( y(0) \) satisfies \( \frac{1}{2} < y(0) < 1 \)?

\( y_1(0) = 1 \) and \( y_2(0) = \frac{1}{2} \), so

\[ y_2(0) < y(0) < y_1(0) \]

By the Uniqueness Theorem, solution curves don’t cross, so \( y_2(t) < y(t) < y_1(t) \) for all \( t \).

Either from this or the phase line, we can see that \( y \to 0 \) as \( t \to \infty \).

Moreover, since \( y_2(t) \to \infty \) as \( t \to -2^- \), we can tell that \( y \) becomes undefined for some \( t = -2 \).
4. (a) Use Euler's method with $\Delta t = 1$ to approximate the solution of the initial value problem $\frac{dy}{dt} = (2-y)(y+1)$, $y(0) = 1$ at $t = 3$.

\[
t_{k+1} = t_k + \Delta t, \quad y_{k+1} = y_k + \Delta t \cdot (2-y_k)(y_k+1)
\]

$t_0 = 0 \quad y_0 = 1$
\[y_1 = 1 + 1 \cdot (2-1)(1+1) = 3\]
$t_1 = 1 \quad y_2 = 3 + 1 \cdot (2-3)(1+3) = -1$
\[y_2 = 3 + 1 \cdot (2-3)(1+3) = -1\]
$t_2 = 2 \quad y_3 = -1 + 1 \cdot (2+1)(-1+1) = -1$
\[y_3 = -1 + 1 \cdot (2+1)(-1+1) = -1\]

(b) Does your approximation seem reasonable? Why or why not?

No, because it jumps back and forth across the equilibrium at 2 and settles at a different equilibrium.

In fact we know the correct solution approaches 2 (from the phase line).