1. (35 points total) Consider the system $\frac{dY}{dt} = AY$, where $A = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$.

(a) (10 points) Find the eigenvalues of $A$.

Solution: The characteristic polynomial is $\lambda^2 - 2\lambda + 2 = 0$, so the eigenvalues are

$$\lambda = \frac{2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

(b) (5 points) Determine from the eigenvalues alone what type of equilibrium the system has at the origin.

Solution: Since the eigenvalues are complex with a positive real part, the system has a **spiral source** at the origin.
(c) (10 points) Find the general solution of the system.

**Solution:** We first find an eigenvector to go with the complex eigenvalues $1 + i$: the matrix equation

$$
\begin{pmatrix}
0 - (1 + i) & -2 \\
1 & 2 - (1 + i)
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
-1 - i & -2 \\
1 & 1 - i
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$
gives us the equation (from the second component, which is simpler)

$$x + (1 - i)y = 0,$$

from which we conclude that $\begin{pmatrix}
-1 + i \\
1
\end{pmatrix}$ is an eigenvector.

Remember you can double-check this:

$$
\begin{pmatrix}
0 & -2 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
-1 + i \\
1
\end{pmatrix}
= 
\begin{pmatrix}
-2 \\
1 + i
\end{pmatrix}
= (1 + i)
\begin{pmatrix}
-1 + i \\
1
\end{pmatrix}
\checkmark
$$

From this eigenvalue and eigenvector we get a complex solution

$$Y_c(t) = e^{(1+i)t}
\begin{pmatrix}
-1 + i \\
1
\end{pmatrix}
= e^t (\cos t + i \sin t)
\begin{pmatrix}
-1 + i \\
1
\end{pmatrix}
= e^t \left( -\cos t - \sin t \cos t + ie^t \left( \cos t - \sin t \sin t \right) \right),$$

from which we get the general real solution

$$Y(t) = k_1 e^t \begin{pmatrix}
-\cos t - \sin t \\
\cos t
\end{pmatrix}
+ k_2 e^t \begin{pmatrix}
\cos t - \sin t \\
\sin t
\end{pmatrix}.$$

(d) (10 points) Find the particular solution with initial condition $Y(0) = (1, 0)$.

**Solution:** Plugging $t = 0$ into the above solution, we get

$$k_1 \begin{pmatrix}
-1 \\
1
\end{pmatrix}
+ k_2 \begin{pmatrix}
1 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0
\end{pmatrix}.$$ 

Obviously $k_1 = 0$, $k_2 = 1$ works, so the desired solution is

$$Y(t) = e^t \begin{pmatrix}
\cos t - \sin t \\
\sin t
\end{pmatrix}.$$
2. (30 points total) Consider the system \( \frac{dY}{dt} = AY \), where \( A = \begin{pmatrix} 2 & 4 \\ 3 & -5 \end{pmatrix} \).

(a) (10 points) Find the eigenvalues of \( A \).

The characteristic polynomial is \( \lambda^2 - T \lambda + D = \lambda^2 + 3 \lambda - 22 \), so the eigenvalues are
\[
\lambda = \frac{-3 \pm \sqrt{3^2 + 4 \cdot 22}}{2} = \frac{-3 \pm \sqrt{97}}{2}.
\]
For reference below, notice that since \( 97 > 9 \), \( \sqrt{97} > 3 \), and so
\[
\frac{-3 + \sqrt{97}}{2} > 0 \quad \text{and} \quad \frac{-3 - \sqrt{97}}{2} < 0.
\]

(b) (10 points) Find the eigenvectors of \( A \).

For \( \lambda = \frac{-3 + \sqrt{97}}{2} \),
\[
\begin{pmatrix} 2 - \left( \frac{-3 + \sqrt{97}}{2} \right) & 4 \\ 3 & -5 - \left( \frac{-3 + \sqrt{97}}{2} \right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7 - \sqrt{97}}{2} & 4 \\ 3 & \frac{-7 - \sqrt{97}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
gives the equation \( \frac{7 - \sqrt{97}}{2} x + 4 y = 0 \), leading to (for example) \( \begin{pmatrix} 4 \\ \frac{-7 - \sqrt{97}}{2} \end{pmatrix} \).
For reference below, notice that this vector points into the first quadrant.

For \( \lambda = \frac{-3 - \sqrt{97}}{2} \),
\[
\begin{pmatrix} 2 - \left( \frac{-3 - \sqrt{97}}{2} \right) & 4 \\ 3 & -5 - \left( \frac{-3 - \sqrt{97}}{2} \right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7 + \sqrt{97}}{2} & 4 \\ 3 & \frac{-7 + \sqrt{97}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
gives the equation \( \frac{7 + \sqrt{97}}{2} x + 4 y = 0 \), leading to (for example) \( \begin{pmatrix} \frac{-4}{\sqrt{97}} \\ \frac{7 + \sqrt{97}}{2} \end{pmatrix} \).
For reference below, notice that this vector points into the second quadrant.
(c) (10 points) Sketch the phase portrait of the system. Include all straight-line solutions as well as solution curves passing through each of the points \((1, 0), (0, 1), (-1, 0),\) and \((0, -1)\). Indicate directions on all solution curves.

\textbf{Solution:} From the above information, the system has a saddle equilibrium, with outward-pointing straight-line solutions in the first and third quadrants and inward-pointing straight-line solutions in the second and fourth quadrants. This results in a similar phase portrait to problem 3(e) on the sample exam.
3. (20 points) Solve the initial value problem

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = (1, 0).$$

**Solution:** The characteristic polynomial is $\lambda^2 - T\lambda + D = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, so $1$ is a repeated eigenvalue.

The general solution is therefore

$$\mathbf{Y}(t) = e^t \mathbf{V}_0 + t e^t \left( \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{V}_0$$

$$= e^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^t \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= e^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^t \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}.$$

Plugging in $x_0 = 1, y_0 = 0$, we get

$$\mathbf{Y}(t) = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
4. (15 points) Consider the one-parameter family of linear systems \( \frac{dY}{dt} = \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix} Y \).

Identify which types of behaviors the system exhibits for which values of \( a \).

**Solution:** We first compute \( T = 2 \) and \( D = 1 - a \). So this system corresponds to a point on the vertical line \( T = 2 \) in the \( T-D \) plane. There are two thresholds where the behavior of the system changes:

- \( D = 0 \), corresponding to \( a = 1 \).
- \( D = \frac{1}{4} T^2 \), corresponding to \( 1 - a = 1 \), or \( a = 0 \).

So the relevant ranges are:

- \( a < 0 \): Here \( D > 1 = \frac{1}{4} T^2 \), and \( T > 0 \), so the system has a **spiral source** at the origin.
- \( a = 0 \): Here \( D = 1 = \frac{1}{4} T^2 \), and \( T > 0 \), so the system has a **source** at the origin, with only a single straight-line solution.
- \( 0 < a < 1 \): Here \( 0 < D < 1 = \frac{1}{4} T^2 \), and \( T > 0 \), so the system has a **source** at the origin.
- \( a = 1 \): Here \( D = 0 \), so the system has a line of equilibrium points.
- \( a > 1 \): Here \( D < 0 \), so the system has a **saddle** at the origin.