1. (10 points) Consider the two following systems of differential equations:

A:
\[
\frac{dx}{dt} = x + y \\
\frac{dy}{dt} = y^2 - 1
\]

B:
\[
\frac{dx}{dt} = x^2 - 1 \\
\frac{dy}{dt} = y - x
\]

Identify which direction field is for which system, and explain your answer.

**Solution:** Field I is for system B and field II is for system A.

There are many possible ways to tell the difference. One is that for system A, \( \frac{dx}{dt} > 0 \) when \( x = 0 \) and \( y > 0 \), which is consistent with II but not I; whereas for system B, \( \frac{dx}{dt} < 0 \) for \( x = 0 \), which is consistent with I but not II.
2. (20 points) Solve the initial value problem

\[ \frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 8y = 0, \quad y(0) = 2, \quad y'(0) = 0. \]

Solution: We try a solution of the form \( y(t) = e^{st} \), and get the characteristic equation

\[ s^2 + 6s + 8 = 0, \]

with solutions \( s = -2 \) and \( s = -4 \). With the linearity principle we get a general solution of the form

\[ y(t) = k_1 e^{-2t} + k_2 e^{-4t}. \]

Therefore

\[ y'(t) = -2k_1 e^{-2t} - 4k_2 e^{-4t}, \]

so our initial conditions tell us

\[
\begin{align*}
  k_1 + k_2 &= 2 \\
  -2k_1 - 4k_2 &= 0.
\end{align*}
\]

The second equation tells us \( k_1 = -2k_2 \), and substituting this into the first equation gives us \( -k_2 = 2 \). Therefore \( k_2 = -2 \) and \( k_1 = 4 \) (Check this!), giving us the solution

\[ y(t) = 4e^{-2t} - 2e^{-4t}. \]

(Check this!)
3. (20 points total) Consider the following variation of the SIR model in which recovereds gradually become susceptible again (e.g., because of mutations in the infection).

\[
\begin{align*}
\frac{dS}{dt} &= -SI + 0.5R \\
\frac{dI}{dt} &= SI - 0.5I \\
\frac{dR}{dt} &= 0.5I - 0.5R
\end{align*}
\]

(a) (5 points) Derive a system in the two dependent variables $S$ and $I$ using the fact that $R = 1 - (S + I)$.

**Solution:** The third equation follows from the first two, so we can ignore it, and by substituting $R = 1 - (S + I)$ in the first equation, we get

\[
\frac{dS}{dt} = -SI + 0.5 - 0.5S - 0.5I
\]

\[
\frac{dI}{dt} = SI - 0.5I
\]

(b) (15 points) Show that there is an equilibrium point of this system with $S_0 > 0$ and $I_0 > 0$. What does this say about the long-term behavior of the infection?

**Solution:** For $\frac{dI}{dt} = 0$ with $I > 0$ we need $S = 0.5$. Substituting this in the first equation, to have $\frac{dS}{dt} = 0$ we need

\[-0.5I + 0.5 - 0.25 - 0.5I = 0,
\]

which simplifies to $I = 0.25$. So $S_0 = 0.5$, $I_0 = 0.25$ is an equilibrium point of this system.

This means that the system predicts it is possible for the infection to persist in 25% of the population indefinitely. (That is, at any given time some 25% of the population is infected; not that the same 25% of the population remains infected.)

Note that we *don't* know, without more work, whether solution curves approach this equilibrium point or not. So we *can’t* say — yet — whether the system predicts that if we start with a larger or smaller infected population, then it will tend toward 25% of the total. (We’ll get to how we could predict that later in the semester.)
4. (40 points) Consider the system $\frac{dY}{dt} = AY$, where $A = \begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix}$.

(a) (10 points) Find the eigenvalues of $A$.

**Solution:** The characteristic polynomial is

$$\lambda^2 - (-1 + 5)\lambda + (-1 \cdot 5 - (-4)2) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3),$$

so the eigenvalues are 1 and 3.

(b) (5 points) Determine from the eigenvalues alone the type of equilibrium this system has at the origin.

**Solution:** Since both eigenvalues are positive, the equilibrium is a source.

(c) (10 points) Find eigenvectors of $A$ corresponding to each of the eigenvalues.

**Solution:** For $\lambda = 1$, we need

$$\begin{pmatrix} -2 & 2 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so $x = y$. Picking $x = 1$, we get an eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Remember you can check this:

$$\begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 + 2 \\ -4 + 5 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \checkmark$$

For $\lambda = 3$, we need

$$\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so $-4x + 2y = 0$, so $y = 2x$. Picking $x = 1$, we get an eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Again, remember you can check this!
(d) (5 points) Write the general solution of the system.

Solution:

\[ Y(t) = k_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \]

(e) (10 points) Sketch the phase portrait of the system. Include all straight-line solutions as well as solution curves in each sector bounded by straight-line solutions. Indicate directions on all solution curves.

Solution: