A partition of a positive integer $n$ is a way of writing $n$ as a sum of a nonincreasing sequence of positive integers. For example,

\[
4 = 4 \\
= 3 + 1 \\
= 2 + 2 \\
= 2 + 1 + 1 \\
= 1 + 1 + 1 + 1
\]
gives all partitions of 4. We write $P(n)$ for the number of partitions of $n$.

Suppose $p_1, \ldots, p_k$ are distinct primes and $n_1, \ldots, n_k \geq 1$. Prove that, up to isomorphism, there are exactly $P(n_1) \cdots P(n_k)$ distinct groups of order $p_1^{n_1} \cdots p_k^{n_k}$. 