1. Assume that $G$ is finite and $\text{char } F \not| |G|$. Let $V$ be an $F(G)$-module, and let $P : V \to V$ be an $F$-linear projection onto an $F(G)$-submodule $W \subseteq V$ (i.e., $P^2 = P$ and $P(V) = W$.) Prove that the map $Q : V \to V$ defined by

$$Q(v) = |G|^{-1} \sum_{g \in G} gP(g^{-1}v)$$

is a projection onto $W$ and is an $F(G)$-module homomorphism.

2. If $f : G_1 \to G_2$ is a groups homomorphism and $\rho : G_2 \to GL(V)$ is a representation of $G_2$, the pullback of $\rho$ by $f$ is the representation

$$f^*\rho := \rho \circ f : G_1 \to GL(V)$$

of $G_1$. Prove that if $f$ is surjective, then $f^*\rho$ is irreducible if and only if $\rho$ is irreducible.

3. Let $H < G$ and let $\rho : G \to GL(V)$ be a representation of $G$.

(a) Show that if $\rho|_H$ is an irreducible representation of $H$, then $\rho$ is irreducible.

(b) Show that the subspace

$$V = \{(x_1, x_2, x_3) \in \mathbb{C}^3 | x_1 + x_2 + x_3 = 0\}$$

gives an irreducible subrepresentation of the standard representation of $S_3$.

(c) Show that the converse of part (a) is false.

4. Determine all irreducible finite dimensional representations of the group $\mathbb{Z}_p$ over the field $\mathbb{F}_p$.  