1. (a) Suppose that $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ has distinct eigenvalues. Prove that $\Lambda$ commutes with $B \in M_n$ if and only if $B$ is diagonal.

(b) Suppose that $A \in M_n$ has $n$ distinct eigenvalues. Prove that if $B \in M_n$ commutes with $A$, then $A$ and $B$ are simultaneously diagonalizable.

2. The spectral radius of $A \in M_n$ is

$$\rho(A) = \max\{ |\lambda| : \lambda \in \sigma(A) \},$$

where $\sigma(A)$ is the spectrum of $A$, that is, the set of eigenvalues of $A$.

Prove that if $A, B$ commute, then $\rho(A + B) \leq \rho(A) + \rho(B)$ and $\rho(AB) \leq \rho(A)\rho(B)$.

3. Show that if $A \in M_n$ is upper triangular and normal, then $A$ is diagonal.

4. Let $A \in M_n$ have eigenvalues $\lambda_1, \ldots, \lambda_n$, listed with (algebraic) multiplicity.

   (a) Prove that for each $k$,

   $$\sum_{j=1}^{n} \lambda_j^k = \text{tr } A^k.$$

   (b) Prove that

   $$\sum_{j=1}^{n} |\lambda_j|^2 \leq \|A\|^2_F,$$

   with equality if and only if $A$ is normal.