1. Show that if $A \in M_n$ is doubly stochastic, then $\sigma_1(A) = 1$.

2. Let $x, y \in \mathbb{R}^n$. Prove that
$$x^\perp + y^\perp \prec x + y \prec x^\perp + y^\perp.$$  

*Hint:* Instead of proving this directly, use Fan and Lidskii’s majorization theorems.

3. (a) Show that if $V$ is any inner product space and $\|\cdot\|$ is the norm induced by the inner product on $V$, then
$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$ (1)
for each $x, y \in V$.

(b) Prove that for $p \neq 2$, the $\ell_p$ norm on $\mathbb{C}^n$ is not induced by any inner product. (Note it’s not enough to check only that it doesn’t come from the usual inner product.)

*Remark:* The identity (1) is called the parallelogram identity. It is also true that if a norm satisfies (1) for each $x, y \in V$ then that norm is induced by an inner product; see problem 5.1.P12 in the textbook for an outline of a proof.