1. Consider the matrix \( A = \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix} \). Compute each of the following explicitly for \( k \geq 1 \), and discuss the behavior as \( k \to \infty \).

   (a) \( A^k \)
   
   (b) \( \rho(A^k) \)
   
   (c) \( \|A^k\|_{1 \to 1} \)
   
   (d) \( \|A^k\|_{\infty \to \infty} \)
   
   (e) \( \|A^k\|_{2 \to 2} \)

2. (a) Prove that if \( A \in M_n \) is normal, then \( \rho(A) = \|A\|_{2 \to 2} \) (the spectral norm of \( A \)).

   (b) Prove that the spectral radius \( \rho \) is not submultiplicative on \( M_n \) — that is, there exist \( A \) and \( B \) with \( \rho(AB) > \rho(A)\rho(B) \).

3. Let \( A \in M_n \) have singular values \( \sigma_1 \geq \cdots \geq \sigma_n \). Prove that if \( \lambda \) is any eigenvalue of \( A \), then \( \sigma_n \leq |\lambda| \leq \sigma_1 \).

4. Suppose \( A \in M_n \) is nonsingular, \( B \in M_n \) is singular, and \( \|\cdot\| \) is a submultiplicative norm on \( M_n \). Prove that

\[
\|A - B\| \geq \frac{1}{\|A^{-1}\|}.
\]

Thus there is a limit to how closely a given nonsingular matrix can be approximated by singular matrices.

**Hint:** Since

\[
A^{-1}B = A^{-1}[A - (A - B)] = I_n - A^{-1}(A - B)
\]

is singular, what can you say about \( A^{-1}(A - B) = I_n - A^{-1}B \)?