1. (a) Prove that if $A \in M_n$ and $\langle Ax, x \rangle$ is real for every $x \in \mathbb{C}^n$, then $A$ is Hermitian. Thus we can simplify the definition of positive (semi)definite matrices by dropping the explicit requirement that $A$ be Hermitian.

(b) Give an example of a matrix $A \in M_n(\mathbb{R})$ such that $\langle Ax, x \rangle \geq 0$ for every $x \in \mathbb{R}^n$, but $A$ is not positive semidefinite.

2. (a) Give an example of a positive definite matrix which has some negative matrix entries.

(b) Give an example of a Hermitian matrix which has only positive matrix entries but which is not positive semidefinite.

3. Let $A \in M_n$ be defined by $a_{jk} = \min\{j, k\}$. Prove that $A$ is positive definite in two ways:

(a) by finding an explicit Cholesky factorization, and

(b) via Sylvester’s criterion.

4. Let $D \in H_{n+1}(\mathbb{R})$ be a symmetric real matrix, indexed by $0, 1, \ldots, n$, such that $d_{ii} = 0$ for all $i$ and $d_{ij} > 0$ for all $i \neq j$. Define $A \in H_n(\mathbb{R})$ by

$$a_{ij} = d_{0i}^2 + d_{0j}^2 - d_{ij}^2.$$ 

Prove that there exist $x_0, x_1, \ldots, x_n \in \mathbb{R}^n$ with $\|x_i - x_j\|_2 = d_{ij}$ if and only if $A$ is positive semidefinite.

**Hint:** For the “only if” direction, consider the Gram matrix of the vectors $v_i = x_i - x_0$. For the “if” direction, mimic the proof that a positive semidefinite matrix is a Gram matrix and define $x_0 = 0$. 