Math 405 Homework
April 21, 2016

1. Let $\mu$ and $\nu$ be probability distributions on $\{1, \ldots, n\}$. (That is, $\mu, \nu \in M_{1,n}$, $\mu, \nu \geq 0$, and $\sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \nu_j = 1$.) The total variation distance between $\mu$ and $\nu$ is

$$TV(\mu, \nu) = \max \left\{ \left| \sum_{j \in J} \mu_j - \sum_{j \in J} \nu_j \right| : J \subseteq \{1, \ldots, n\} \right\}.$$ 

That is, it is the largest possible difference between the probabilities of being in a given set.

Show that $TV(\mu, \nu) = \frac{1}{2} \|\mu - \nu\|_1$.

2. Let $P > 0$ be a stochastic matrix and $\pi > 0$ be any positive probability distribution. Define a matrix $Q$ by

$$q_{ij} = \begin{cases} p_{ij} \min \left\{ 1, \frac{\pi_j p_{ji}}{\pi_i p_{ij}} \right\} & \text{if } i \neq j, \\ 1 - \sum_{k \neq i} p_{ik} \min \left\{ 1, \frac{\pi_k p_{ki}}{\pi_i p_{ik}} \right\} & \text{if } i = j. \end{cases}$$

Show that $Q$ is a stochastic matrix which is reversible with respect to $\pi$.

Remarks: The positivity assumptions are only for convenience and can be weakened.

The construction of the stochastic matrix $Q$ is called the Metropolis–Hastings algorithm, and is an important tool in applications for constructing a Markov chain with a desired stationary distribution.