On the comparison of volumes of quantum states

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To determine the separability and entanglement is a hard problem!

What is the relative size of the set of separable and entangled quantum states within the set of quantum states in terms of some measure?
Complex Hilbert space $\mathcal{H} = \mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2} \cdots \otimes \mathbb{C}^{D_n}$ with complex dimension $N = D_1 \cdots D_n$. Each factor of $\mathcal{H}$ system corresponds to a subsystem of $\mathcal{H}$. 

The set of quantum states may be identified as the set of density matrices $\mathcal{D}$:

$$\mathcal{D} = \{ \rho : N \times N \text{ positive definite matrix with trace 1, i.e., } \text{tr} \rho = 1 \}$$

The dimension of $\mathcal{D}$ is $d = N^2 - 1$. 

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Separable and Entangled Quantum states

Separable quantum states

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$$S = \text{conv}\{\rho_1 \otimes \cdots \otimes \rho_n, \rho_i \in \mathcal{D}(\mathbb{C}^{D_i})\}.$$
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What is the probability of $$S$$ and $$\mathcal{E}$$ in $$\mathcal{D}$$?
Peres-Horodecki Positive Partial Transpose Criterion

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**Peres-Horodecki PPT Criterion**

Let $\mathcal{PPT} = \{\rho \in \mathcal{D}(\mathcal{H}) : \text{s.t. } T_1(\rho) \geq 0\}$. Then

$$S \subset \mathcal{PPT} \subset \mathcal{D}.$$ 

That is, a separable state must satisfy positive partial transpose criterion. Equivalently, a non-PPT state must be entangled.
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How precise is the Peres-Horodecki PPT criterion (as a tool to detect the separability)?
Measures $dV$ on $D$ in literature have common features, such as,

\[ dV = d\nu \times d\gamma, \]

where $d\nu$ is some measure on the chamber of the simplex (of eigenvalues)
\[ \{ (\lambda_1, \cdots, \lambda_N) : \sum_i \lambda_i = 1, \lambda_i \geq 0 \} \]
with order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, and $d\gamma$ is the measure on the flag manifold (of eigenvectors)
\[ \mathbb{F}_N = U(N)/[U(1)] \]
\[ d\gamma = 1 \cdots \prod_{i < j} 2 \text{Re}(U_{ij} dU_{ij}) \text{Im}(U_{ij} dU_{ij}) \]
where $dU$ is the variation of unitary matrix $U$ such that $U + dU$ is also an unitary matrix.
Measures $dV$ on $\mathcal{D}$ in literature have common features, such as,

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Measures $dV$ on $\mathcal{D}$ in literature have common features, such as, $dV = d\nu \times d\gamma$, where $d\nu$ is some measure on $\Delta_1$: the chamber of the simplex (of eigenvalues) $\{(\lambda_1, \cdots, \lambda_N): \sum_i \lambda_i = 1, \lambda_i \geq 0\}$ with order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, and $d\gamma$ is the measure on the flag manifold $F_N = U(N)/[U(1)]_N$: $d\gamma = 1 \cdots N \prod_{i < j} 2\Re(U_i - 1)dU_{ij} \Im(U_i - 1)dU_{ij}$, where $dU$ is the variation of unitary matrix $U$ such that $U + dU$ is also an unitary matrix.
Measures $dV$ on $\mathcal{D}$ in literature have common features, such as, 
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where $d\nu$ is some measure on $\Delta_1$: the chamber of the simplex (of eigenvalues) \( \{ (\lambda_1, \cdots, \lambda_N) : \sum_i \lambda_i = 1, \lambda_i \geq 0 \} \) with order \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \), and $d\gamma$ is the measure on the flag manifold (of eigenvectors) \( \mathcal{F}^N = \mathcal{U}(N)/[\mathcal{U}(1)]^N \):
\[ d\gamma = \prod_{i<j}^{1 \cdots N} 2\text{Re}(U^{-1}dU)_{ij} \text{Im}(U^{-1}dU)_{ij} \]
where $dU$ is the variation of unitary matrix $U$ such that $U + dU$ is also an unitary matrix.
The Hilbert-Schmidt measure:

\[ dV_{HS} = \sqrt{N} \prod_{i<j}^{1\cdots N} (\lambda_i - \lambda_j)^2 \delta_0(\sum_i \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \, d\gamma; \]
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- Bures measure:

\[ dV_B = \frac{2^{\frac{2-N-N^2}{2}}}{\sqrt{\lambda_1 \cdots \lambda_N}} \prod_{i<j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} \delta_0(\sum \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \; d\gamma; \]
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\]

Induced measure by partial trace on \( \mathcal{H}_N \otimes \mathcal{H}_K \) \((K \geq N)\):

\[
dV_{N,K} = \prod_{i=1}^N \lambda_i^{K-N} \prod_{i<j} (\lambda_i - \lambda_j)^2 \delta_0(\sum \lambda_i - 1) \prod_{i=1}^N d\lambda_i d\gamma;
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\]

the \( \alpha \)-volume \( (\alpha > 0) \):
\[
dV_{\alpha} = \prod_{i=1}^{N} \lambda_i^{\alpha-1} \prod_{i<j} (\lambda_i - \lambda_j)^2 \delta_0(\sum_i \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \, d\gamma.
\]
Measure induced by Metric on \( \mathcal{D} \)

\( dV_{HS} \) is induced by Hilbert-Schmidt distance

\[
d_{HS}(\rho, \sigma) = \|\rho - \sigma\|_2 = \sqrt{tr(\rho - \sigma)^\dagger(\rho - \sigma)}.
\]
Measure induced by Metric on $\mathcal{D}$

$dV_{HS}$ is induced by Hilbert-Schmidt distance

$$d_{HS}(\rho, \sigma) = \|\rho - \sigma\|_2 = \sqrt{\text{tr}(\rho - \sigma)^\dagger (\rho - \sigma)}.$$  

$dV_B$ is induced by Bures distance

$$d_B(\rho, \sigma) = \sup_{\{E_i\}} \left( \sum_i \left[ \sqrt{\text{tr}(E_i \rho)} - \sqrt{\text{tr}(E_i \sigma)} \right]^2 \right)^{\frac{1}{2}}$$

where the supremum runs over all the POVM (positive operator valued measurement) $\{E_i\}$, that is, for some $1 \leq k \leq N$,

$$\sum_{i=1}^k E_i = \text{Id}_N, \text{ and } E_i = E_i^\dagger, E_i \geq 0, \ i = 1, \cdots, k.$$
Let $K \geq N$, and $\rho$ be a density matrix on $\mathcal{H}_N \otimes \mathcal{H}_K$, then

$$\rho = \begin{pmatrix}
A_{11} & \cdots & A_{1N} \\
\vdots & \ddots & \vdots \\
A_{N1} & \cdots & A_{NN}
\end{pmatrix}.$$ 

Define the partial trace over $\mathcal{H}_K$ as

$$\rho^A = Tr_B(\rho), \text{ where } (\rho^A)_{ij} = tr(A_{ij}) \text{ for } i, j = 1, \cdots, N.$$
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The partial trace process allows us to view states on $\mathcal{H}_N$ as a (pure) state on (much higher) dimensional space $\mathcal{H}_N \otimes \mathcal{H}_K$. Then the measures induced by partial trace may be considered as a projection of the $(NK - 1)$ dimensional simplex of eigenvalues into simplex of $(N - 1)$ dimension.
Comparison of $\alpha$-volume with Hilbert-Schmidt volume

Let $\mathcal{K}$ be a measurable subset of $\mathcal{D}$ (on $\mathcal{H}_N$), and $d = N^2 - 1$. 

Let $\alpha > 0$ be a (fixed) constant, $\alpha_{\text{max}} = \max\{1, \alpha\}$, and $\alpha_{\text{min}} = \min\{1, \alpha\}$. There exist universal constants $c_1, C_1 > 0$, s.t.,

$$c_1 \frac{\text{VR}_{\text{HS}}(\mathcal{K}, \mathcal{D})}{\alpha_{\text{max}} \exp\left((1 - \alpha_{\text{max}}) \ln \ln\left(e/\xi\right) N^2 - 1\right)} \leq \frac{\text{VR}_{\alpha}(\mathcal{K}, \mathcal{D})}{\alpha_{\text{min}} \exp\left((1 - \alpha_{\text{min}}) \ln \ln\left(e/\xi\right) N^2 - 1\right)} \leq C_1 \frac{\text{VR}_{\text{HS}}(\mathcal{K}, \mathcal{D})}{\alpha_{\text{min}}}.$$
Comparison of $\alpha$-volume with Hilbert-Schmidt volume

Let $\mathcal{K}$ be a measurable subset of $\mathcal{D}$ (on $\mathcal{H}_N$), and $d = N^2 - 1$. The $\alpha$-volume radii ratio of $\mathcal{K}$ to $\mathcal{D}$ is

$$VR_{\alpha}(\mathcal{K}, \mathcal{D}) = \left( \frac{V_{\alpha}(\mathcal{K})}{V_{\alpha}(\mathcal{D})} \right)^{1/d}$$

and the Hilbert-Schmidt volume radii ratio of $\mathcal{K}$ to $\mathcal{D}$ is

$$\xi = VR_{HS}(\mathcal{K}, \mathcal{D}) = VR_1(\mathcal{K}, \mathcal{D}).$$
Comparison of $\alpha$-volume with Hilbert-Schmidt volume

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Let $\alpha > 0$ be a (fixed) constant, $\alpha_{\text{max}} = \max\{1, \alpha\}$, and $\alpha_{\text{min}} = \min\{1, \alpha\}$. There exist universal constants $c_1, C_1 > 0$, s.t.,

$$c_1 VR_{HS}(\mathcal{K}, \mathcal{D})^{\alpha_{\text{max}}} \exp\left(\frac{(1 - \alpha_{\text{max}}) \ln \ln(e/\xi)}{N^2 - 1}\right) \leq VR_\alpha(\mathcal{K}, \mathcal{D})$$

$$\leq C_1 VR_{HS}(\mathcal{K}, \mathcal{D})^{\alpha_{\text{min}}} \exp\left(\frac{(1 - \alpha_{\text{min}}) \ln \ln(e/\xi)}{N^2 - 1}\right).$$
Comparison of $\alpha$-volume with Bures volume

Let $\zeta = \text{VR}_\alpha(\mathcal{K}, \mathcal{D})$, and the Bures volume radii ratio of $\mathcal{K}$ to $\mathcal{D}$ be

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$$\text{VR}_B(\mathcal{K}, \mathcal{D}) = \left(\frac{V_B(\mathcal{K})}{V_B(\mathcal{D})}\right)^{1/d}.$$


There exist universal constants $c_2, C_2 > 0$, such that

$$c_2 \min\{1, \frac{1}{2\alpha}\} \exp \left(\frac{(1 - \max\{1, \frac{1}{2\alpha}\}) \ln \ln(\frac{e}{\zeta})}{N^2 - 1}\right) \leq \text{VR}_B(\mathcal{K}, \mathcal{D}) \leq C_2 \min\{\frac{1}{2}, \frac{1}{2\alpha}\} \exp \left(\frac{\ln \ln(\frac{e}{\zeta})}{2N}\right).$$

- $\alpha = 1$: D. Ye, (Journal of Mathematical Physics (JMP), 2009).
Large number of small subsystems

Let $\mathcal{H} = (\mathbb{C}^D)^\otimes n$ with (small) $D$, and $\alpha_D = \frac{\log_D(1+\frac{1}{D})}{2} - \frac{\log_D(D+1)}{2D^2}$. 
Large number of small subsystems

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There exist universal constants \( c_3, c_3', C_3, C_3' > 0 \), s.t., for \( \alpha > 0 \),

\[
\frac{c_3}{N^{1/2+\alpha_D}} \leq VR_B(S, D) \leq \frac{c_3}{N^{1/2+\alpha_D}} \sqrt{\frac{(Dn \ln n)^{1/2}}{N^{1/2+\alpha_D} \max\{1, \alpha\}}} \leq VR_\alpha(S, D) \leq \frac{c_3'}{N^{1/2+\alpha_D}} \min\{1, \alpha\} \left( \frac{(Dn \ln n)^{1/2}}{N^{1/2+\alpha_D}} \right)^{\min\{1, \alpha\}}.
\]

G. Aubrun and S. J. Szarek proved:

\[
\tilde{c}_3 \frac{N^{1/2+\alpha_D}}{N^{1/2+\alpha_D}} \leq VR_HS(S, D) \leq \tilde{c}_3 \frac{(Dn \ln n)^{1/2}}{N^{1/2+\alpha_D}}.
\]
Small number of large subsystems

Let $\mathcal{H} = (\mathbb{C}^D)^n$ with (small) $n$.


There exist universal constants $c_4, c'_4, C_4, C'_4 > 0$, s.t., for $\alpha > 0$,

$$\frac{c_4^n}{N^{1/2-1/(2n)}} \leq \text{VR}_B(S, D) \leq C_4 \sqrt{\frac{(n \ln n)^{1/2}}{N^{1/2-1/(2n)}}}.$$

$$\left(\frac{c'_4^n}{N^{1/2-1/(2n)}}\right)^{\max\{1, \alpha\}} \leq \text{VR}_{\alpha}(S, D) \leq C'_4 \left(\frac{(n \ln n)^{1/2}}{N^{1/2-1/(2n)}}\right)^{\min\{1, \alpha\}}.$$

G. Aubrun and S. J. Szarek proved:

$$\frac{\tilde{c}_4^n}{N^{1/2-1/(2n)}} \leq \text{VR}_{HS}(S, D) \leq \tilde{C}_4 \frac{(n \ln n)^{1/2}}{N^{1/2-1/(2n)}}.$$
There exist absolute constants $c_0, c'_0 > 0$ such that for any bipartite system $\mathcal{H} = \mathbb{C}^D \otimes \mathbb{C}^D$, 

$$c_0 \leq \text{VR}_B(\text{PPT}, \mathcal{D}) \leq 1;$$
$$c'_0 \leq \text{VR}_\alpha(\text{PPT}, \mathcal{D}) \leq 1.$$

G. Aubrun and S. J. Szarek proved: $\tilde{c}_0 \leq \text{VR}_{HS}(\text{PPT}, \mathcal{D}) \leq 1$. 

Peres-Horodecki PPT becoming imprecise as a tool to detect separability for large $N$


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G. Aubrun and S. J. Szarek proved: $\tilde{c}_0 \leq \text{VR}_{HS}(\text{PPT}, \mathcal{D}) \leq 1$.

Conclusion: $\text{VR}_B(S, \text{PPT})$ and $\text{VR}_\alpha(S, \text{PPT})$ go to 0, hence, the Peres-Horodecki PPT criterion is not precise as a tool to detect separability for large $N$. 
Let $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}$. The $\alpha$-probability of $S$ in $\mathcal{D}$ is defined as

$$\mathbb{P}_\alpha(S, n, D) =: \frac{V_\alpha(S)}{V_\alpha(\mathcal{D})}.$$ 

\begin{align*}
\mathbb{P}_2(S, 8, 2) &\leq 2.1 \times 10^{-1595}, & \mathbb{P}_2(S, 5, 3) &\leq 1.52 \times 10^{-5301}, \\
\mathbb{P}_{0.5}(S, 8, 2) &\leq 8.8 \times 10^{-479}, & \mathbb{P}_{0.5}(S, 5, 3) &\leq 9.5 \times 10^{-2351}.
\end{align*}

The conditional $\alpha$-probability of $S$ given PPT is

$$\mathbb{P}_\alpha(S|\text{PPT}, n, D) =: \frac{V_\alpha(S)}{V_\alpha(\text{PPT})}.$$ 

\begin{align*}
\mathbb{P}_{1.1}(S|\text{PPT}, 12, 2) &\leq 2.5 \times 10^{-2721940}, & \mathbb{P}_{1.1}(S|\text{PPT}, 8, 3) &\leq 1.82 \times 10^{-12248770}.
\end{align*}


Thank you!