Chair’s Challenge, Fall, 1999

The Chair of the Department will contribute $25.00 of his own hard earned salary to Math Club member(s) for correct solutions of the three problems below.

Rules:

1. Prize will be awarded for correct solutions to all three problems. Entries are to be written (in good form), and submitted to advisors of the Math Club. In the event of multiple correct entries, a tie-breaker will be devised. Entries will be due later in the term; a precise date will be set later by the advisors.

2. The Math Club advisors are the judges. Decisions of the judges is final.

3. Collaboration is permitted. Teams may submit entries and split the prize.

4. You may consult other sources and quote results—please document with citations.

1. For a function $g(x)$, defined and continuous for $x \geq 0$, define inductively $g_0(x) = g(x)$, $g_{n+1}(x) = \int_0^x g_n(t) \, dt$.
   (a) Prove that the series $\sum_{n=0}^{\infty} g_n(x)$ converges to a continuous function $G(x)$ on $[0, \infty)$.
   (b) Characterize the set of $G(x)$ that can be expressed as such a series.
   (Suggestion: try first the case $g(x) = 1$.)

2. The number 381654729 is an interesting number. It is composed of the digits 1 through 9, each used once, and has the following property: the number composed of the first $k$ digits is (evenly) divisible by $k$ for $k = 1, \ldots, 9$, i.e., 1 divides 3 (trivially), 2 divides 38, 3 divides 381, etc. Problem: find all 9-digit numbers composed of the digits 1 through 9, each used once, with this property.

3. Consider a cube of side $s$ that is rapidly spun around a long diagonal (running from one corner to the opposite corner) as rotation axis. It will appear to fill out a solid (you can see this with a real cube—position the cube vertically on a surface, lightly hold the top point, and spin it like a top). Compute the volume of that solid.