Lectures on Elastic Curves and Rods

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Abstract. These five lectures constitute a tutorial on the Euler elastica and the Kirchhoff elastic rod. We consider the classical variational problem in Euclidean space and its generalization to Riemannian manifolds. We describe both the Lagrangian and the Hamiltonian formulation of the rod, with the goal of examining the (Liouville-Arnol'd) integrability. We are particularly interested in determining closed (i.e., periodic) solutions.

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1. CLASSICAL BERNOULLI–EULER ELASTICA

The classical curve known as the *elastica* is the solution to a variational problem proposed by Daniel Bernoulli to Leonhard Euler in 1744, that of minimizing the bending energy of a thin inextensible wire (See, e.g., [27], §263). The mathematical idealization of this problem is that of minimizing the integral of the squared curvature for curves of a fixed length satisfying given first order boundary data. In this lecture, we will use the classical techniques of the calculus of variations to derive the equations of the elastica.

Consider regular curves (curves with nonvanishing velocity vector) in Euclidean space defined on a fixed interval $[a_1,a_2]$:

$$\gamma : [a_1,a_2] \longrightarrow \mathbb{R}^3 \quad \left\|\gamma'(t)\right\| = \frac{ds}{dt} = v \neq 0$$

We will assume (for technical reasons) that the (geodesic) curvature k of γ is nonvanishing. This will allow us to define the *Frenet Frame* along the curve.

The Frenet frame $\{T,N,B\}$ is orthonormal and satisfies

$$\gamma' = vT \quad \frac{dT}{ds} = kN \quad \frac{dN}{ds} = -kT + \tau B \quad \frac{dB}{ds} = -\tau N$$

The elastica minimizes the bending energy

$$\mathscr{F}(X) = \int_\gamma k(s)^2 ds = \int_{a_1}^{a_2} k(t)^2\; vdt$$

with fixed length and boundary conditions. Accordingly, let α_1 and α_2 be points in $\mathbb{R}^3$ and α_1' and α_2' nonzero vectors.

We will consider the space of smooth curves

$$\Omega = \{\gamma | \gamma(a_i) = \alpha_i, \gamma'(a_1) = \alpha_i'\}$$

and the subspace of unit-speed curves

$$\Omega_u = \{\gamma \in \Omega | \left\|\gamma'\right\| \equiv 1\}$$

Later on we need to pay more attention to the precise level of differentiability of curves, but we will ignore that for now.

$\mathscr{F}^\lambda : \Omega \longrightarrow \mathbb{R}$ is defined by

$$\mathscr{F}^\lambda(\gamma) = \frac{1}{2}\int_\gamma \left\|\gamma''\right\|^2 + \Lambda(t)\left(\left\|\gamma'\right\|^2 - 1\right) dt$$

One version of the Lagrange multiplier principle says a minimum of $\mathscr{F}$ on Ω_u is a stationary point for $\mathscr{F}^\lambda$ for some $\Lambda(t)$. ($\Lambda(t)$ is a *pointwise* multiplier, constraining speed.) The name $\mathscr{F}^\lambda$ for the function will be justified later, when we will see that $\Lambda(t)$ depends on a constant λ.

Assume that γ is an extremum of $\mathscr{F}^\lambda$. Then if W is a vector field along γ, that is, an infinitesimal variation of the curve, we have

$$\partial\mathscr{F}^\lambda(W) = \frac{\partial}{\partial\varepsilon}\mathscr{F}^\lambda(\gamma+\varepsilon W)|_{\varepsilon=0} = 0$$

$$0 = \frac{1}{2}\frac{\partial}{\partial\varepsilon}\int_{a_1}^{a_2} \left\|(\gamma+\varepsilon W)''\right\|^2 + \Lambda\left\|(\gamma+\varepsilon W)'\right\|^2 - \Lambda(t)dt$$

$$= \int_{a_1}^{a_2} \gamma''\cdot W'' + \Lambda(t)\gamma'\cdot W' dt$$

Integrating by parts,

$$0 = \int_{a_1}^{a_2} -\gamma'''\cdot W' - (\Lambda\gamma')'\cdot W dt + (\gamma''\cdot W' + \Lambda\gamma'\cdot W)\Big|_{a_1}^{a_2}$$

$$= \int_{a_1}^{a_2} \left[\gamma'''' - (\Lambda\gamma')'\right]\cdot W dt$$

$$+ (\gamma''\cdot W' + (\Lambda\gamma' - \gamma''')\cdot W)\Big|_{a_1}^{a_2}$$

$$= \int_{a_1}^{a_2} E(\gamma)\cdot W dt + (\gamma''\cdot W' + (\Lambda\gamma' - \gamma''')\cdot W)\Big|_{a_1}^{a_2}$$

$$\text{where } E(\gamma) = \gamma'''' - \frac{d}{dt}(\Lambda\gamma')$$

The elastica must satisfy

$$E(\gamma) = \gamma'''' - \frac{d}{dt}(\Lambda\gamma') \equiv 0$$

for some function $\Lambda(t)$.

Integrating,

$$\gamma''' - \Lambda(t)\gamma' \equiv J$$

for J a constant vector.

This equation can also be derived from Noether's Theorem: If γ is a solution curve and W is an infinitesimal symmetry, then $\gamma'' \cdot W' + (\Lambda\gamma' - \gamma''') \cdot W$ is constant. In particular, for a translational symmetry, W is constant; so

$$(\Lambda\gamma' - \gamma''') \cdot W = \text{const.}$$

Letting W range over all translations, we get

$$\Lambda\gamma' - \gamma''' = C$$

for C some constant field.

Now it is helpful to assume γ is parametrized by arclength s. Then $\gamma' = \gamma_s = T$, $\prod \gamma'' = kN, \gamma''' = -k^2T + k_sN + k\tau B$, so

$$\gamma''' - \Lambda(s)\gamma' = (-k^2 - \Lambda(s))T + k_sN + \tau B = J$$

Differentiate J to get

$$0 = J_s = (-3kk_s - \Lambda_s)T + (k_{ss} - k^3 - \Lambda k - k\tau^2)N + (k\tau_s + 2k_s\tau)B$$

From this it follows that $\Lambda(s) = -\frac{3}{2}k^2 + \frac{\lambda}{2}$ for some constant λ.

The vector field $J(s) = \frac{k^2-\lambda}{2}T + k_sN + k\tau B$ is constant along the curve. Thus it is the restriction of a translation field to the curve. From $J_s = 0$ we get the equations

$$k_{ss} + \frac{1}{2}k^3 - k\tau^2 - \frac{\lambda k}{2} = 0 \tag{1}$$

and

$$k\tau_s + 2k_s\tau = 0 \tag{2}$$

We may use Noether's theorem to derive additional first integrals of these equations. Recall that for any variation W,

$$0 = \int_{a_1}^{a_2} E(\gamma) \cdot W ds + (\gamma'' \cdot W' - J \cdot W)\Big|_{a_1}^{a_2}$$

If W is a symmetry, we have

$$\gamma'' \cdot W' - J \cdot W = \text{constant}$$

Now let W be the restriction of a rotation field:

$$W = \gamma \times W_0, \quad W_0' = 0$$

Differentiating,

$$kN \cdot T \times W_0 - J \cdot \gamma \times W_0 = \text{const}$$

or

$$(kN \times T - J \times \gamma) \cdot W_0 = \text{const}$$

Since this works for any W_0, we get

$$kB + J \times \gamma = A$$

for A a constant vector. So the vector field

$$I = kB = A + \gamma \times J$$

is the restriction of an isometry to the curve.

The vector fields J and I play an important role in the integration of the equations of the elastica. A *Killing field* along a curve is a vector field along the curve which is the restriction of an infinitesimal isometry of the ambient space. If γ is an elastica in $\mathbb{R}^3$, then we have two Killing fields along γ:

$$J = \frac{k^2 - \lambda}{2} T + k_s N + k\tau B$$

and

$$I = kB = A + \gamma \times J$$

where A and J are constant fields.

$$k^2\tau = I \cdot J = A \cdot J = c \tag{3}$$

is constant, as is

$$4\,\|J\|^2 = (k^2 - \lambda)^2 + 4k_s^2 + 4k^2\tau^2 = a^2 \tag{4}$$

Observe that equation 1 integrates to equation 4 and equation 2 integrates to equation 3. Now we can integrate these equations as follows:

Eliminating τ and replacing k^2 by u:

$$(u - \lambda)^2 + \frac{(u_s)^2}{u} + \frac{4c^2}{u} = a^2$$

or

$$(u_s)^2 = P(u)$$

for P a cubic polynomial.

Solving this differential equation gives

$$k^2 = u = k_0^2 \left(1 - \frac{p^2}{w^2} \operatorname{sn}^2\left(\frac{k_0}{2w} s, p\right)\right) = \frac{c}{\tau} \tag{5}$$

With $\operatorname{sn}(x, p)$ the *elliptic sine with parameter* p, and with $p, w,$ and k_0 parameters.

The parameters $p, w,$ and k_0 are related to the constants λ and c by

$$2\lambda = \frac{k_0^2}{w^2}(3w^2 - p^2 - 1)$$

$$4c^2 = \frac{k_0^6}{w^4}(1-w^2)(w^2-p^2)$$

A planar curve has $\tau = 0$, so $c = 0$. Thus either $w = 1$ or $w = p$. The parameter k_0 determines the maximum curvature.

Up to similarity, every solution corresponds to a point in the triangle $0 \leq p \leq w \leq 1$. The planar curves correspond to two of the three edges of the triangle. The third edge of the triangle, $p = 0$ is made up of curves of constant curvature and torsion (helices).

What do the three-dimensional solutions look like? We can use the Killing fields to construct a preferred cylindrical coordinate system. Choose coordinates in $\mathbb{R}^3$ so that

$$J = \begin{pmatrix} 0 \\ 0 \\ \frac{a}{2} \end{pmatrix}, \quad A = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, \quad b \geq 0$$

The second equation is achieved by replacing γ by $\gamma - K$ for some K (translation).
Since $I = \gamma \times J + A$, we have

$$c = I \cdot J = A \cdot J = \frac{ab}{2}$$

$$I - \frac{I \cdot J}{J \cdot J} J = I - A = \gamma \times J$$

We have the coordinate fields

$$\partial z = \frac{2}{a} J$$

$$\partial \theta = \frac{2}{a} \gamma \times J = \frac{2}{a}(I - \frac{4c}{a^2} J)$$

$$\partial r = \frac{\partial z \times \partial \theta}{\|\partial z \times \partial \theta\|} = \frac{J \times B}{\|J \times B\|}$$

Writing $T = \frac{dr}{ds}\partial r + \frac{dz}{ds}\partial z + \frac{d\theta}{ds}\partial \theta$, we get

$$\frac{dr}{ds} = T \cdot \partial r \quad \frac{dz}{ds} = T \cdot \partial z \quad \frac{d\theta}{ds} = \frac{T \cdot \partial \theta}{\|\partial \theta\|^2}$$

and the equations for γ can be integrated explicitly.

Using equation 4, the differential equation for r can be seen to be

$$\frac{dr}{ds} = \frac{2k_s}{\sqrt{4k_s^2 + (k^2 - \lambda)^2}} = \frac{2kk_s}{\sqrt{a^2k^2 - 4c^2}}$$

This integrates to

$$r = \frac{2}{a^2}\sqrt{a^2k^2 - 4c^2}$$

So r has the same periodicity and critical points as k. The elastica lies between two concentric cylinders (the inner one perhaps degenerating to a line) around the z - axis.

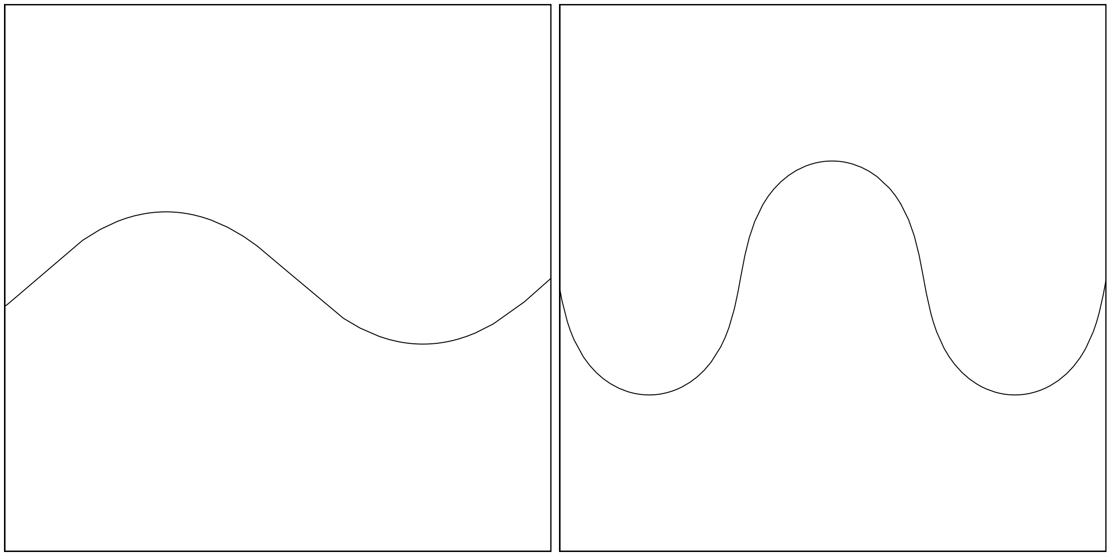

FIGURE 1. Wavelike elastica

the maxima of the curvature occur on the outer cylinder and the minima on the inner cylinder.

In the 2-dimensional case ($c = 0$), the curve lies in a strip parallel to the $z-axis$. In this case the formula for r, the distance from the axis, simplifies to

$$r = \frac{k}{a}$$

When $w = p$, $k^2 = k_0^2\left(1 - \mathrm{sn}^2\left(\frac{k_0}{2p}s, p\right)\right)$ so

$$k = k_0\,\mathrm{cn}\left(\frac{k_0}{2p}s, p\right)$$

The curvature oscillates between $-k_0$ and $+k_0$. We call such a curve a "wavelike" elastica. (Figure 1 and Figure 2)

When $w = 1$, $k^2 = k_0^2\left(1 - p^2\,\mathrm{sn}^2\left(\frac{k_0}{2}s, p\right)\right)$ so

$$k = k_0\,\mathrm{dn}\left(\frac{k_0}{2}s, p\right)$$

(where dn(x) is the elliptic delta), and k is non-vanishing. We call such a curve "orbit-like". (Figure 3a)

The borderline case, $p = w = 1$, (Figure 3b) has non-periodic curvature:

$$k = k_0 \mathrm{sech}\frac{k_0}{2}s$$

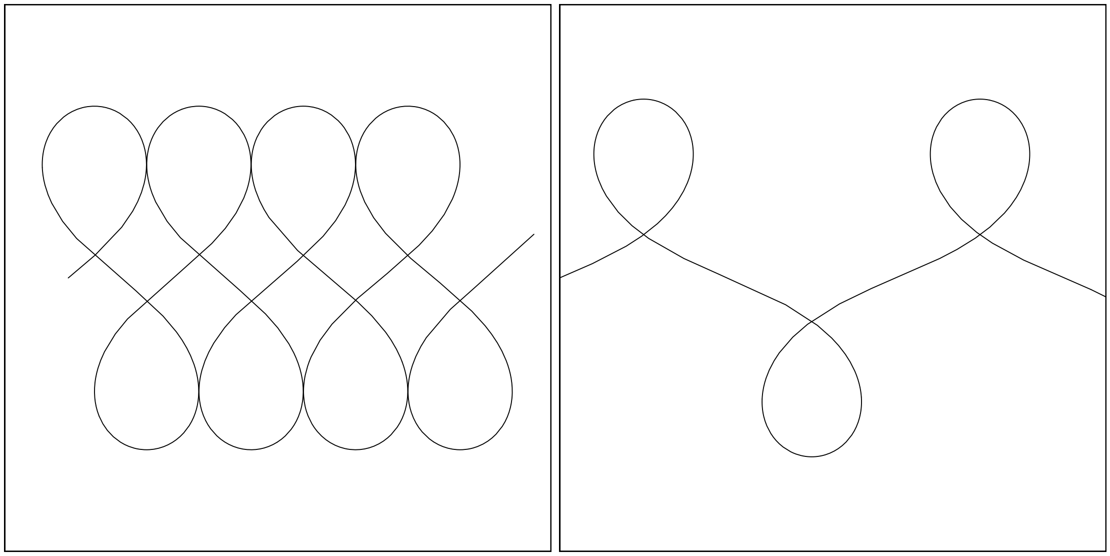

FIGURE 2. Wavelike elastica

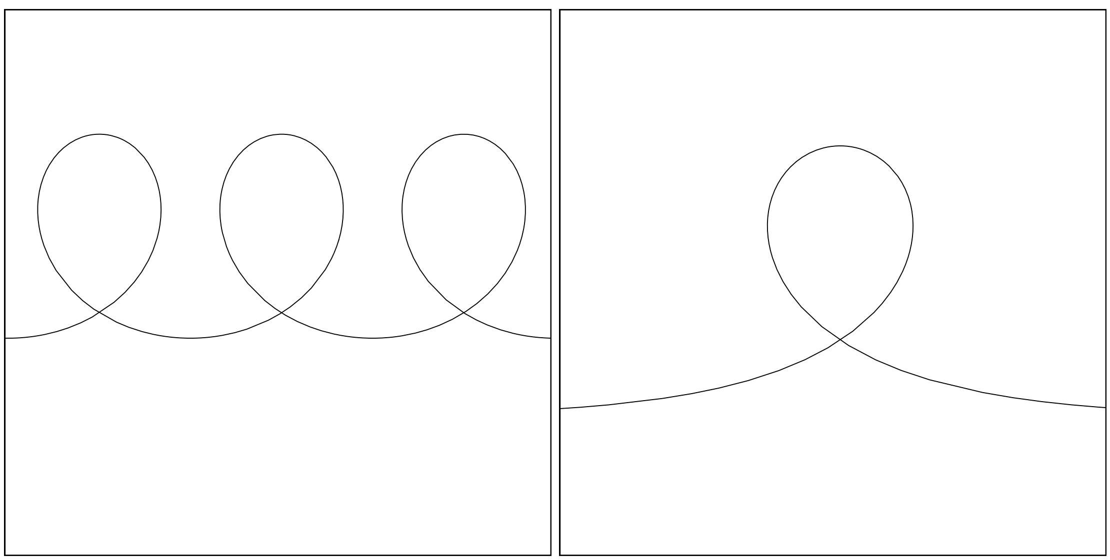

FIGURE 3. a) Orbitlike elastica b) Borderline elastica

For $p = 0$ we get the helices:

$$k \equiv k_0 \qquad \tau \equiv \tau_0 = \frac{c}{k_0^2}.$$

The curves with $0 < p < w < 1$ are non-planar. For such curves, $0 < k < k_0$ and the curve has non-vanishing curvature and torsion.

When is the curve *closed*? Its curvature (and its torsion) is always periodic except in the case of a borderline elastic curve ($p = 1$), which is obviously not closed. We need the coordinates of γ to be periodic.

To answer this we must briefly review the properties of elliptic functions. (See [6] for exhaustive details.) The complete elliptic integrals $E(p)$ and $K(p)$ are given by:

$$K(p) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-p^2\sin^2\phi}}\, d\phi$$

and

$$E(p) = \int_0^{\frac{\pi}{2}} \sqrt{1-p^2\sin^2\phi}\, d\phi$$

The elliptic function $\operatorname{sn}(x,p)$ is an odd function with $\operatorname{sn}(x+2K(p),p) = -\operatorname{sn}(x,p)$. So the curvature of an elastica is periodic with period $2wK(p)/k_0$.

If $\triangle z$ and $\triangle\theta$ represent the change in z and θ over one period of k, then γ is a smooth closed curve if and only if $\triangle z = 0$ and $\triangle\theta$ is rationally related to 2π.

$$\begin{aligned}\triangle z &= \int_0^{\frac{2w}{k_0}K(p)} \langle \partial z, T\rangle\, ds = \frac{2}{a}\int_0^{\frac{2w}{k_0}K(p)} (k^2-\lambda)ds \\ &= \frac{4w}{ak_0}\int_0^{K(p)} (k_0^2-\lambda) - k_0^2\frac{p^2}{w^2}\operatorname{sn}^2(x,p)\, dx\end{aligned}$$

The closure condition may be written:

$$\triangle z = 0 \Longleftrightarrow 1 + w^2 - p^2 - 2\frac{E(p)}{K(p)} = 0$$

There is one closed planar curve (besides the circle): It requires

$$w = p \quad 2E(p) = K(p) \Longleftrightarrow p \approx .82$$

We have seen that J is a constant field, the restriction of a translation field to the curve. From the formula

$$J = \frac{k^2-\lambda}{2}T + k'N + k\tau B$$

It is clear that the curvature achieves its extrema at places where $N \cdot J = 0$.

The second condition for closure is that the θ coordinate be periodic. Let $\triangle\theta$ denote the increase in the θ coordinate in one period of the curvature function. Then it is necessary that $\triangle\theta$ be a rational multiple of 2π. The formula for $\triangle\theta$ in terms of the parameters p and w is quite complicated; see [19] for details. The essential fact is that as one the point on the curve $\triangle z$ varies, the value of $\triangle\theta$ varies monotonically between 0 and $-\pi$. It therefore takes on every rational multiple of pi between those two values. This leads to the following theorem:

Theorem 1 (Langer and Singer, 1983 [19]) *$\triangle\theta$ is monotonically decreasing from π to 0 along $\triangle z = 0$. Thus there are infinitely many closed elastic curves which are nonplanar. All such elastica are embedded, lie on embedded tori of revolution, and represent (m,n) - torus knots, one for each $m > 2n$.*

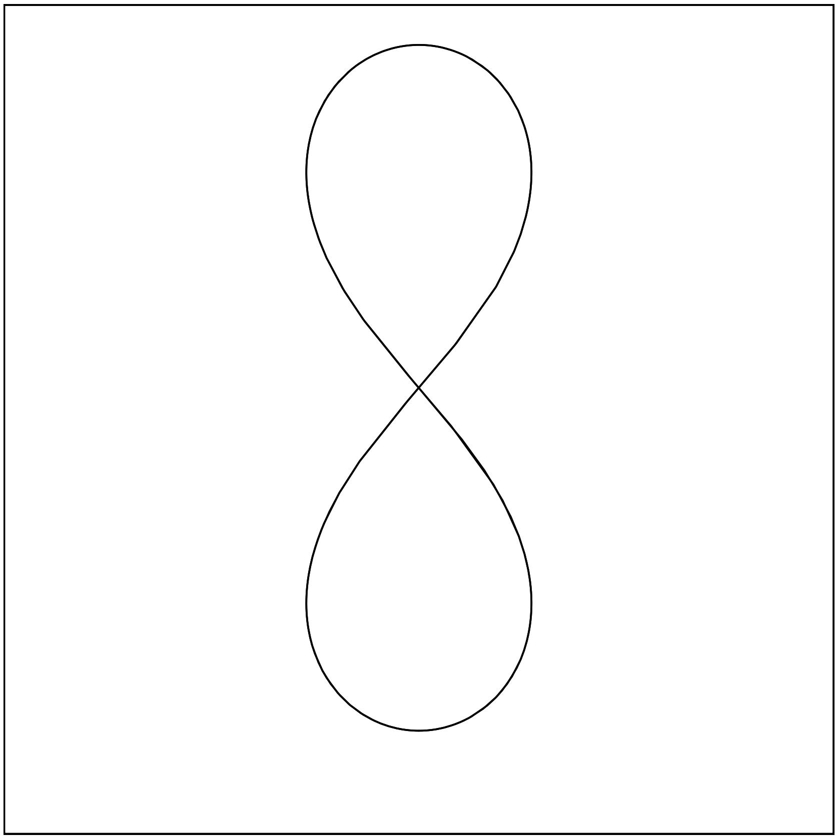

FIGURE 4. The figure-eight elastica

Figure 5 allows one to visualize the space of elastic curves in $\mathbb{R}^3$. Three parameters control the size and shape of an elastica. One parameter, k_0, gives the maximum curvature of the curve; dilations of $\mathbb{R}^3$ will adjust this. So we are reduced to two parameters, p and w, with $0 \leq p \leq w \leq 1$, and the parameter space is a triangle. Two of the three sides of the triangle correspond to planar elastic curves, while the third side corresponds to helices. The three vertices of the triangle correspond to borderline elastica, circle, and straight line. The interior of the triangle consists entirely of nonplanar curves. The curve $\triangle z = 0$ contains all of the closed elastic curves. The line $p^2 + w^2 = 1$ contains those (nonplanar) elastic curves that cross the axis of symmetry. Finally, there is a line along which are the solutions of the *free elastica* problem: for these curves there is no length constraint. We will have more to say about the free elastica later.

2. THE ELASTICA IN A RIEMANNIAN MANIFOLD

In this lecture we will formulate a generalized variational problem, that of the elastica in a Riemannian manifold. By this we mean a curve which is an extremal for the integral of the squared (geodesic) curvature among curves with specified boundary conditions. Here we summarize the machinery needed for calculations.

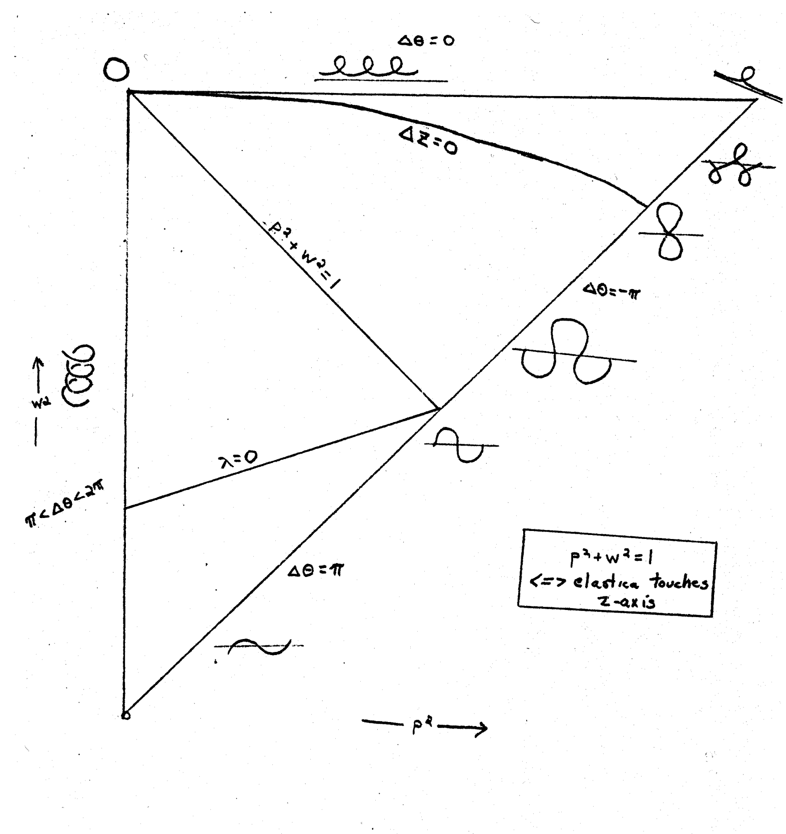

FIGURE 5. Parameter space of elastica in $\mathbb{R}^3$

In what follows, M is a smooth Riemannian manifold, with Riemannian metric $g(X,Y) =< X,Y >$, that is, a positive-definite symmetric bilinear form on tangent vectors X and Y at each point. The ordinary derivative is replaced by the covariant derivative $\nabla_X Y$, which measures the derivative of a vector *field* Y in the direction of a vector X.

For vector fields X and Y the equality of mixed partial derivatives is replaced by the bracket formula:

$$\nabla_X Y - \nabla_Y X = [X,Y] = XY - YX$$

Ex: If $X = \frac{\partial}{\partial x}$, and $Y = x\frac{\partial}{\partial y}$ in local coordinates, then

$$\nabla_X Y - \nabla_Y X = \frac{\partial}{\partial y} - 0 = \left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right]$$

$\gamma(t)$ is an immersed curve in M then it has velocity vector $V = vT$ and squared geodesic curvature

$$k^2 = \|\nabla_T T\|^2$$

The Frenet equations for a curve are

$$\gamma' = vT \quad \nabla_T T = kN \quad \nabla_T N = -kT + \tau B \quad \nabla_T B = -\tau N \tag{6}$$

For a family of curves $\gamma_w(t) = \gamma(w,t)$ we will write

$$W = W(w,t) = \frac{\partial \gamma}{\partial w}$$

$$V(w,t) = \frac{\partial \gamma}{\partial t} = v(w,t)T(w,t)$$

So V is velocity and $v = \frac{ds}{dt}$ is speed, and W represents an infinitesimal variation of the curve. s is the arclength parameter along a curve.

The basic formulas needed in calculating the Euler equations are as follows:

$$0 = [W,V] = [W,vT] = W(v)T + v[W,T] \tag{7}$$

So $[W,T] = -\frac{W(v)}{v}T = gT$.

$$2vW(v) = W(v^2) = 2\langle \nabla_W V, V \rangle = 2\langle \nabla_V W, V \rangle = 2v^2 \langle \nabla_T W, T \rangle \tag{8}$$

So $W(v) = -gv,\ g = -\langle \nabla_T W, T \rangle$.

$$W(k^2) = 2\langle \nabla_T \nabla_T W, \nabla_T T \rangle + 4gk^2 + 2\langle R(W,T)T, \nabla_T T \rangle \tag{9}$$

Here the curvature tensor R is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Proof of (9):

$$\begin{aligned} W(k^2) &= 2\langle \nabla_W \nabla_T T, \nabla_T T \rangle \\ &= 2\left\langle \nabla_T \nabla_W T + \nabla_{[W,T]} T + R(W,T)T, \nabla_T T \right\rangle \\ &= 2\left\langle \nabla_T \nabla_T W + \nabla_T (gT) + \nabla_{gT} T \right. \\ &\quad \left. + R(W,T)T, \nabla_T T \right\rangle \\ &= 2\langle \nabla_T \nabla_T W, \nabla_T T \rangle + 2\langle R(W,T)T, \nabla_T T \rangle \\ &\quad + 4g\langle \nabla_T T, \nabla_T T \rangle \end{aligned}$$

In what follows, $\gamma : [0,1] \to M$ is a curve of length L. Now for fixed constant λ let

$$\mathscr{F}^\lambda(\gamma) = \frac{1}{2}\int_0^L k^2 + \lambda ds = \frac{1}{2}\left(\int_0^L k^2 ds + \lambda L\right)$$

$$= \frac{1}{2}\int_0^1 (\|\nabla_T T\|^2 + \lambda) v(t) dt$$

For a variation γ_w with variation field W we compute

$$\frac{d}{dw}\mathscr{F}^{\lambda}(\gamma_w) = \frac{1}{2}\int_0^1 W(k^2)v + (k^2+\lambda)W(v)dt$$

$$\begin{aligned}
=&\frac{1}{2}\int_0^1 W(k^2) - (k^2+\lambda)g\ ds \\
=&\int_0^1 \langle \nabla_T\nabla_T W, \nabla_T T\rangle + 2gk^2 \\
&+ \langle R(W,T)T, \nabla_T T\rangle - \frac{1}{2}(k^2+\lambda)gds
\end{aligned}$$

One of the symmetries of the curvature tensor allows us to replace $\langle R(W,T)T, \nabla_T T\rangle$ with $\langle R(\nabla_T T, T)T, W\rangle$. Now integrate by parts, using $g = -\langle \nabla_T W, T\rangle$

$$\begin{aligned}
\frac{d}{dw}\mathscr{F}^{\lambda}(\gamma_w) =& \int_0^1 \langle \nabla_T\nabla_T W, \nabla_T T\rangle - \left\langle \nabla_T W, 2k^2 T\right\rangle \\
&+ \langle R(\nabla_T T, T)T, W\rangle + \frac{1}{2}\left\langle \nabla_T W, (k^2+\lambda)T\right\rangle ds \\
=& \int_0^L \langle E, W\rangle ds \\
&+ \left[\langle \nabla_T W, \nabla_T T\rangle + \left\langle W, -(\nabla_T)^2 T + \Lambda T\right\rangle\right]_0^L
\end{aligned}$$

where

$$E = (\nabla_T)^3 T - \nabla_T(\Lambda T) + R(\nabla_T T, T)T$$

and

$$\Lambda = \frac{\lambda - 3k^2}{2}$$

When M is a manifold of constant sectional curvature G, the formula for E can be simplified to

$$E = (\nabla_T)^3 T - \nabla_T(\Lambda_G T)$$

where

$$\Lambda_G = \frac{\lambda - 2G - 3k^2}{2}$$

Now by using the Frenet equations (6) we compute

$$E = \nabla_T\left(\nabla_T kN - \frac{\lambda - 2G - 3k^2}{2}T\right)$$

$$= \nabla_T\left(\frac{k^2 - \lambda + 2G}{2}T + k_s N + k\tau B\right)$$

$$= \frac{2k_{ss} + k^3 - \lambda k + 2Gk - k\tau^2}{2} N + (2k_s\tau + k\tau_s)B$$

The equations $E = 0$ for the elastica become:

$$2k_{ss} + k^3 - \lambda k + 2Gk - k\tau^2 = 0 \tag{10}$$

and

$$2k_s\tau + k\tau_s = 0 \tag{11}$$

The second equation integrates to

$$k^2\tau = c$$

Eliminating τ from the first equation and integrating:

$$k_s^2 + \frac{k^4}{4} + (G - \frac{\lambda}{2})k^2 + \frac{c^2}{k^2} = A$$

Letting $u = k^2$ this becomes

$$u_s^2 + u^3 + 4(G - \frac{\lambda}{2})u^2 - 4Au + 4c^2 = 0$$

This has the following solutions:

1. $u = k^2$=constant, τ=constant (“helices” and circles)
2. $k = k_0 \operatorname{sech}\left(\frac{k_0}{2w}s\right), \tau = 0$ (“borderline elastica”)
3. $k = k_0 \operatorname{cn}\left(\frac{k_0}{2w}s, p\right), \tau = 0$ (“orbitlike elastica”)
4. $k = k_0 \operatorname{dn}\left(\frac{k_0}{2w}s, p\right), \tau = 0$ (“wavelike elastica”)
5. $k^2 = k_0^2\left(1 - \frac{p^2}{w^2}\operatorname{sn}^2\left(\frac{k_0}{2w}s, p\right)\right)$

where $4G - 2\lambda = \frac{k_0^2(1+p^2-3w^2)}{w^2}$ and $0 \le p \le w \le 1$.

2.1. Example: Closed elastic curves on the 2-sphere

The two-sphere $\mathbb{S}^2$ with constant Gauss curvature G provides an interesting and remarkably complex illustration of the theory of elastic curves.[1] Obviously the compactness of the sphere insures that there will be a more interesting collection of closed elastic curves than the plane, but also fact that there are no dilations suggests that the structure

[1] For proofs of results in this subsection, see [21]

is richer. To determine the conditions for closedness, we again rely on the notion of Killing fields.

Proposition 2 *Let M be a (simply-connected) manifold with constant sectional curvature G, and let γ be an elastica in M. Then the vectorfields $J = \frac{k^2-\lambda}{2}T + k_s N + k\tau B$ and $I = kB$ along γ extend to Killing fields (infinitesimal isometries) on M.*

Idea of proof: Verify that when $W = I$ or $W = J$, then W preserves arclength parameter, curvature, and torsion of γ. Since the isometry group of the sphere is three-dimensional, a dimension count shows that vector fields which satisfy these conditions must be the restrictions of Killing fields.

For arclength, one checks that $\langle \nabla_T W, T \rangle = 0$. For curvature, use the formula (9) for $W(k)$. For torsion, use the formula:

$$W(\tau^2) = 2\left\langle \frac{1}{k}(\nabla_T)^3 W - \frac{k_s}{k^2}(\nabla_T)^2 W + \left(\frac{G}{k} + k\right)\nabla_T W - \frac{k_s}{k^2} G W, \tau B \right\rangle$$

In the two-sphere, the Killing field $J = \frac{k^2-\lambda}{2}T + k_s N$ must be the restriction to γ of a rotation field. By choosing coordinates x, y of longitude and latitude on the sphere, we may assume that

$$\frac{\partial}{\partial x} = aJ$$

where a is a constant chosen so that J has unit length on the equator. Since

$$\|J\|^2 = \frac{(k^2-\lambda)^2}{4} + k_s^2 = A + \frac{\lambda^2}{4} - Gk^2$$

it is clear that the norm of J is maximized where k^2 is minimized. So if $k = k_0 \operatorname{cn}\left(\frac{k_0}{2w}s, p\right)$, then k vanishes at the maxima of $\|J\|$. Since $\langle N, J \rangle = k_s \neq 0$ when $k = 0$, the curve γ is transverse to the coordinate curves $y =$ const. at these points. It follows that the curve is crossing the equatorial curve $y = 0$ at the inflection points. The normalizing constant a is precisely $\sqrt{A + \frac{\lambda^2}{4}}$.

Theorem 3 *If γ is a wavelike elastica on a two-dimensional space-form, then the inflection points of γ all lie on a geodesic (the "axis" of the curve).*

A wavelike elastic curve on $\mathbb{S}^2$ is now seen to oscillate across a great circle. Define the *wavelength* Λ of a wavelike elastica to be the amount of progress it makes along its axis in one complete period of k, as measured by arclength along the geodesic. If $\frac{\Lambda}{\pi\sqrt{G}}$ is rational, then the elastica will close up. To determine the closed elastic curves, then, it is necessary to study the dependence of Λ on parameters.

When the parameter λ is fixed at a value less than $2G$, the non-geodesic elastic curves are wavelike, with curvature given by

$$k(s) = k_0 \operatorname{cn}(\frac{k_0 s}{2p}, p)$$

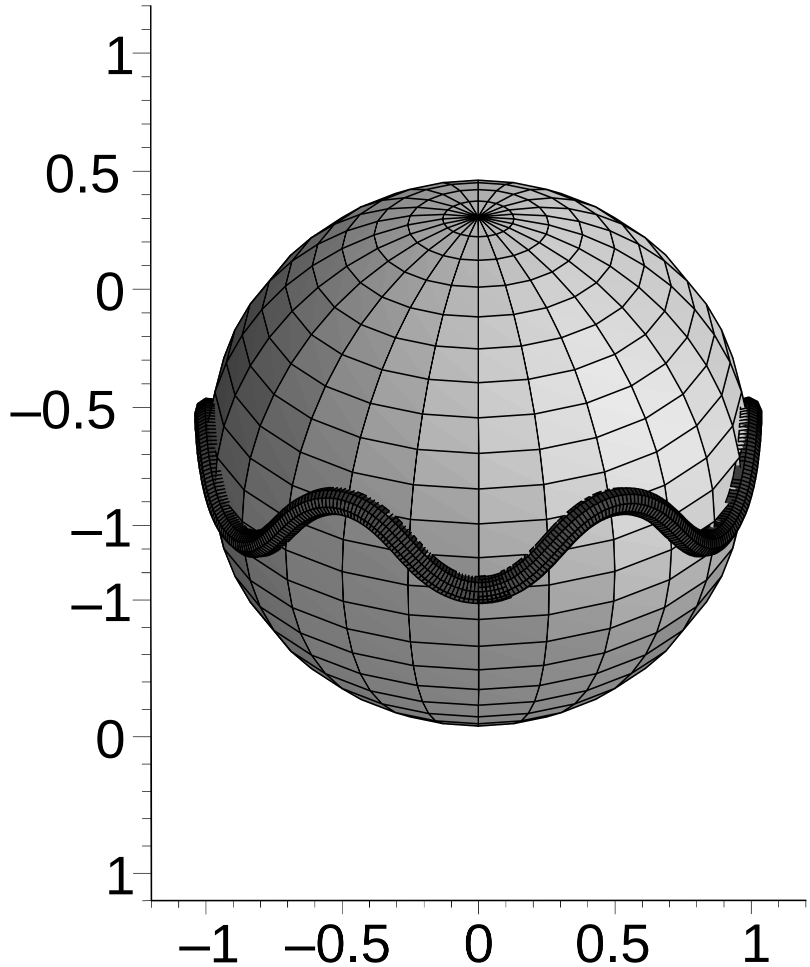

FIGURE 6. A wavelike elastica in $\mathbb{S}^2$

where the maximum curvature is given by

$$k_0 = \frac{p\sqrt{4G-2\lambda}}{\sqrt{1-2p^2}} \quad (0 \le p^2 < \frac{1}{2})$$

The significance of the quantity $4G-2\lambda$ can (partly) be accounted for as follows: By analyzing the *second variation* formula along a closed geodesic, it is possible to determine that the $n-fold$ geodesic is a stable critical point of $\mathscr{F}^\lambda$ if and only if

$$\frac{2n-1}{n^2} \ge \frac{\lambda}{2G}$$

For large λ the length penalty causes the closed geodesic to cease to be a minimum. Indeed, when $\lambda > 2$ the small circle of curvature $\sqrt{\lambda - 2G}$ is a stable equilibrium. At these values, *orbitlike* elastic curves appear, whose curvature is nonvanishing.

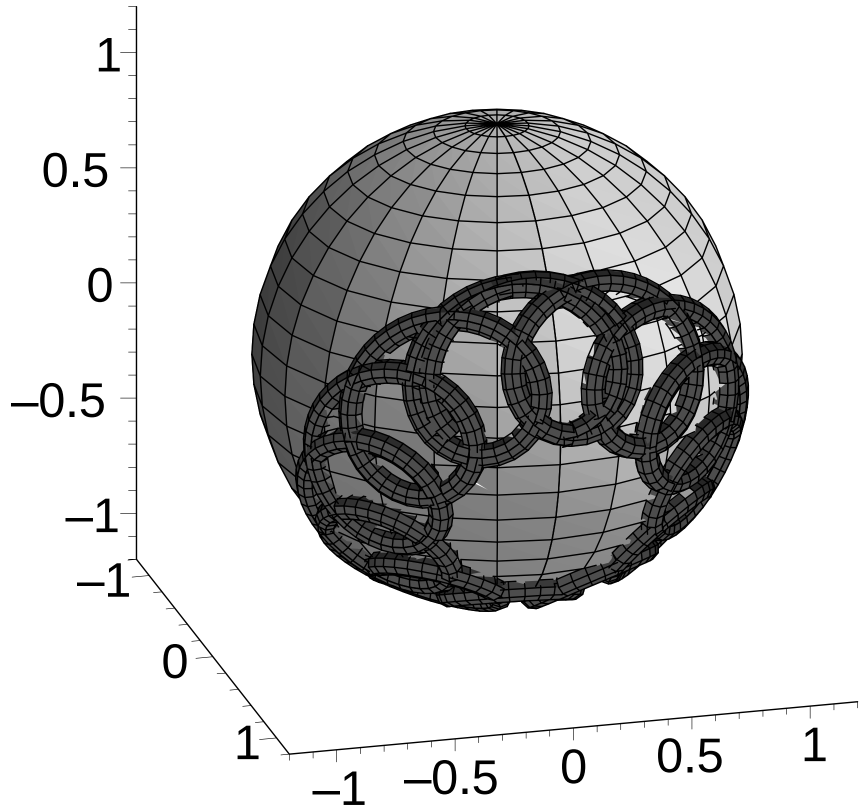

FIGURE 7. An orbitlike elastica in $\mathbb{S}^2$

Now suppose λ is some fixed positive constant less than $2G$. As a specific example, consider $\lambda = 1$. For this value, the equator, the double-covered equator, and the *triple-covered* equator are stable equilibria of $\mathscr{F}^{\lambda}$. As we will see (in lecture 5), the *minimax principle* is valid in this situation. Therefore, since the single and triple equators are regularly homotopic (that is, it is possible to smoothly deform the single circle to the triple circle through regular curves), we expect to find another elastica, a minimax critical point. In fact, using the fact that the regular homotopy can be chosen to preserve a four-fold symmetry, we expect to find an elastica whose wavelength is exactly $\pi\sqrt{G}$, which looks something like the seam on a tennis ball. And indeed, such an elastica does exist.

The full story turns out to be surprisingly subtle. The existence and uniqueness of such minimax critical points turns out to hold for $0 \leq \lambda \leq \frac{8G}{7}$.

Theorem 4 (L-S, 1987 [21]) *Let λ be a fixed constant with $0 \leq \frac{8}{7}G$. Then for each pair of positive integers m,n with*

$$\frac{m}{n} < 1 - \frac{\sqrt{G}}{\sqrt{4G - 2\lambda}}$$

there is a unique elastica $\gamma^{\lambda}_{m,n}$ (up to congruence) which closes up in n periods while traversing the equator m times. All such curves are unstable (minimax) critical points *for* $\mathscr{F}^\lambda$.

3. THE KIRCHHOFF ELASTIC ROD

In the first two lectures we have considered an essentially one-dimensional object, a wire of negligible thickness. In this lecture, we will expand our horizon to include some consideration of the effect of thickness on elastic energy. We consider a thin elastic rod with circular cross-section and uniform density – the uniform symmetric (linear) Kirchhoff rod.[2] For a thorough treatment of the elastic theory of rods, see [1]. The configurations of the rod are described abstractly using *adapted framed curves*:

$$\Gamma = \{\gamma(s); T, M_1, M_2\}$$

where the *centerline* of the rod $\gamma(s)$ is a unit speed curve in $\mathbb{R}^3$, and the *material frame* $(T(s), M_1(s), M_2(s))$ is a positively oriented orthonormal frame. These are related by the condition:

$$\gamma'(s) = T(s)$$

We say that the frame is *adapted* to the curve.

The rotation of the material frame can be described by the *Darboux vector*

$$\Omega = mT - m_2 M_1 + m_1 M_2$$

and the equations

$$\begin{array}{rcccrr} T' & = & \Omega \times T & = & & m_1 M_1 \quad +m_2 M_2 \\ M_1' & = & \Omega \times M_1 & = & -m_1 T & +m M_2 \\ M_2' & = & \Omega \times M_2 & = & -m_2 T & -m M_1 \end{array}$$

The *Total Elastic Energy* of a framed curve is given by

$$E(\Gamma) = \frac{1}{2} \int \underbrace{\alpha_1 (m_1)^2 + \alpha_2 (m_2)^2}_{bending} + \underbrace{\beta m^2}_{twisting} \, ds$$

where α_1, α_2 and β are material constants.

We will be concerned with the *symmetric* case:

$$\alpha = \alpha_1 = \alpha_2$$

In this case, the bending energy is $\frac{\alpha}{2} \int k^2 \, ds$.

To understand the behavior of the rod, we need to define another frame, called the *inertial frame* along a rod by the equations:

$$\begin{array}{rcrr} T' = & & k_2 U & +k_3 V \\ U' = & -k_2 T & & \\ V' = & -k_3 T & & \end{array}$$

---

[2] [23] is the main reference for this section

The inertial frame has no *twist*, i.e., U and V "have no T-component to their angular velocity." That is, the Darboux vector is $\Omega = -k_3 U + k_2 V$. The quantities k_2 and k_3 are the *natural curvatures* of γ. Note that they are related to the geodesic curvature by the equation

$$k^2 = k_2^2 + k_3^2.$$

To more easily understand the inertial frame, define an angle ϕ to be the angle between N and U. That is, write

$$\begin{array}{rcrcr} N &=& U\cos\phi &+& V\sin\phi \\ B &=& -U\sin\phi &+& V\cos\phi \end{array}$$

Differentiating the second equation

$$-\tau N = B' = \phi'(-U\cos\phi - V\sin\phi) + (k_2\sin\phi - k_3\cos\phi)T$$

Comparing, we derive the relationships:

$$\phi' = \tau \quad k_2 = k\cos\phi \quad k_3 = k\sin\phi$$

The natural frame is uniquely determined once its initial value is specified; if we replace ϕ by $\phi+$ const. we get an equivalent frame. Unlike the Frenet frame, the inertial frame only depends on two derivatives of the curve. Since the curvature and torsion are determined by the natural curvatures, the curve itself is uniquely determined up to congruence by its curvatures. The natural frame has the further technical advantage that we need not require k to be nonvanishing.

Now to relate the natural frame to the material frame, assume $U(0) = M_1(0)$. Then we can measure the twisting of the rod by looking at the angle $\theta(s)$ between $M_1(s)$ and $U(s)$.

Write $M_1 = U\cos\theta + V\sin\theta$. Then

$$m = M_1' \cdot M_2 = \theta'(-U\sin\theta + V\cos\theta)\cdot M_2$$

That is,

$$m = \theta'$$

We can describe the rod by $\{\gamma, k_1, k_2, \theta\}$. The energy is

$$E = \frac{1}{2}\int \alpha k^2 + \beta(\theta')^2 \, ds$$

An elastic rod is an equilibrium configuration for the energy with appropriate boundary conditions (usually, having each end fixed in position and clamped.) Note that we can change the rod without changing its centerline by changing θ. We can fix the ends of the rod and its centerline and reduce the energy by minimizing the second term. Thus we have

Proposition 5 *For a rod in equilibrium, θ' is constant. Thus $\theta(s) = ms$, for some fixed constant m.*

If instead of clamping the ends, we held them in collars (so the ends could not change direction but were free to twist), then the energy would be reduced by untwisting until $m = 0$. This shows that an untwisted rod is a minimizer of bending energy – an elastica.

Now assume the rod is closed of length L, and (for convenience) that the material frame M satisfies $M(0) = M(L)$. That is, we take a rod, 'twist' it n times and weld the ends together. The Frenet frame automatically closes up, but the natural frame need not. (More generally, we could assume the angle between $(M(0)$ and $M(L)$ is a fixed value. This would lead to a similar conclusion, except that n would not be an integer.)

Let $\psi = \phi - \theta$ be the angle between the material frame and the Frenet frame. Then our assumption is that $\frac{\psi(L)-\psi(0)}{2\pi}$ is an integer. This leads to:

$$2\pi n = \psi(L) - \psi(0) = \int_0^L \phi'(s) - \theta'(s)ds$$
$$= \int_0^L \tau(s)ds - mL$$

The previous calculation says that an elastic rod centerline has total torsion $\int_0^L \tau(s)ds$ given by the quantity $2\pi n + mL$. Using this, it is possible to formulate a variational problem whose solutions are exactly the elastic rod centerlines.

Theorem 6 *For a curve $\gamma(s)$ define*

$$\mathscr{F}(\gamma) = \lambda_1 \int_\gamma ds + \lambda_2 \int_\gamma \tau ds + \lambda_3 \int_\gamma k^2 ds$$

with k and τ the curvature and torsion and $\lambda_3 \neq 0$. Then an extremal of $\mathscr{F}$ is an elastic rod centerline.

Clearly, when $\lambda_2 = 0$, this is an elastic curve.

For a Kirchhoff elastic rod centerline γ, there are two Killing fields: for constants α and σ:

$$J = \frac{\alpha k^2 - \lambda}{2} T + \alpha k' N + k(\alpha\tau - \sigma)B$$

and

$$I = \sigma T + \alpha k B$$

Theorem 7 *For rod centerline γ there is an 'associated' elastic curve γ_0 whose curvature is k and torsion is $\tau - \frac{\sigma}{2\alpha}$, where k and τ are the curvature and torsion of γ.*

Recall that for elastic curves, There are embedded nonplanar elastic curves, each lying on an embedded torus of revolution,representing an (m,n) - torus knot, one for each $m > 2n$. In the case of rods, there is an extra parameter, allowing for many more closed curves. For rods, the result is:

Theorem 8 (Ivey - Singer, 1998[12]) *Every torus knot type is realized by a smooth closed elastic rod centerline. For any pair of relatively prime positive integers k and* *n there is a one-parameter family of closed elastic rods forming a regular homotopy between the k-times covered circle and the n-times covered circle. The family includes exactly one elastic curve, one self-intersecting elastic rod, and one closed elastic rod with constant torsion.*

4. HAMILTONIAN THEORY

In this lecture, we will investigate the remarkable fact that the equations of the elastica in Euclidean space, and indeed, in space forms, can be completely integrated. The key to understanding this is through the Hamiltonian approach to the variational problems of curves and rods. What follows is a brief survey of the ideas in geometric variational problems, using the machinery of Optimal Control Theory.[3] A different approach to this type of variational problem, using the theory of exterior differential systems, is developed in [5].

We begin with an n-dimensional smooth manifold E, called *Configuration Space*. Let $H : T^*E \to \mathbb{R}$ be a smooth function on the cotangent bundle (or *Phase Space*); H is the *Hamiltonian.*

In canonical coordinates $(p^1,\dots,p^n,q^1,\dots,q^n)$ H defines a time-independent Hamiltonian system:

$$\dot{q}^i = \frac{\partial H}{\partial p^i} \quad \dot{p}^i = -\frac{\partial H}{\partial q^i}$$

However, for certain problems, such as those considered here, a different type of coordinate will be more useful.

If $F,G : T^*E \to \mathbb{R}$, then the *Poisson bracket* is

$$\{F,G\} = \sum_1^n \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i}\frac{\partial G}{\partial q^i}$$

$\{\bullet,H\}$ acts like a derivative:

$$\{FG,H\} = F\{G,H\} + G\{F,H\}$$

In terms of the Poisson bracket, Hamilton's equations can be written:

$$\dot{q}^i = \{q^i,H\} \quad \dot{p}^i = \{p^i,H\} \tag{12}$$

If $\gamma(t) = (p^i(t),q^i(t))$ is a solution curve of equation (12), then we can differentiate any quantity along γ by $\frac{dF}{dt} = \{F,H\}$. In particular, if $\{F,H\} = 0$, then F is a conserved quantity along the integral curves of (12), or a *first integral* of H.

H is *Liouville integrable* if there are functions $F_1,\dots,F_n$ with all $\{F_i,F_j\} = 0, F_n = H$. The F_i are n constants of motion in involution. The Liouville-Arnol'd theorem says that the trajectories of H can be found by quadratures.

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[3] The book [14] by Velimir Jurdjevic is an excellent reference for this lecture. See [13] and [22] for details.

Now let $M = \mathbb{R}^3$, $\mathbb{S}^3$, or $\mathbb{H}^3$. Let $E = FM$ be the space of positively-oriented orthonormal frames $f = (X; f_1, f_2, f_3)$ on M. It is helpful to think of f as a linear map from $\mathbb{R}^3$ to the tangent space at X, taking the standard orthonormal basis (e_1, e_2, e_3) to the orthonormal vectors (f_1, f_2, f_3), where $f_3 = f_1 \times f_2$.

So configuration space E is a 6-dimensional manifold.

The group $SO(3)$ of rotations acts on E on the *right* by rotating frames: If $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation, then $(f, g) \mapsto f \circ g = R_g(f)$. The space E is the total space of a principal fiber bundle over M whose fiber is $SO(3)$.

$$\begin{array}{ccc} E & \longleftarrow & SO(3) \\ \Big\downarrow{\scriptstyle\pi} & & \\ M & & \end{array}$$

The standard basis for the Lie Algebra $\mathfrak{so}(3)$ determines three *fundamental vector fields* on E, as follows:

$$\text{Let } \alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then $[\alpha_1, \alpha_2] = \alpha_3$, $[\alpha_2, \alpha_3] = \alpha_1$, and $[\alpha_3, \alpha_1] = \alpha_2$.

Each α_i gives rise to a one-parameter subgroup of $SO(3)$ and by the right action a one-parameter group of diffeomorphisms of E with the vector field A_i as infinitesimal generator.

The vectors $A_1(f)$,$A_2(f)$,$A_3(f)$ span the *vertical* subspace $\mathbf{V}(f)$ of the tangent space at each point f of E. That is, $\pi_*(A_i(f)) = 0$, and the vectors are linearly independent at each point f.

Note: we may identify E with the Lie Group $\mathscr{G}$ of isometries of M. From that point of view, A_i becomes a *left invariant* vector field.

M	$E \equiv \mathscr{G}$	Matrix description
$\mathbb{R}^3$	$E(3)$	$\begin{pmatrix} 1 & 0 \\ v & R \end{pmatrix}, v \in \mathbb{R}^3, R \in SO(3)$
$\mathbb{S}^3$	$SO(4)$	$A^T A = I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\mathbb{H}^3$	$SO(3,1)$	$A^T J A = J, J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

The Riemannian metric defines the horizontal subspace $\mathbf{H}(f)$ of the tangent space at f. Roughly speaking, given any curve in M passing through $X = \pi(f)$, there is a unique lift of the curve to E achieved by parallel transport of the frame f along the curve. The tangent vector at f to this lift is horizontal. A *basic vectorfield* $B(\xi)$ is a horizontal field (using the Riemannian connection) such that $\pi_*(f)(B(\xi)) = f(\xi)$, where ξ is any vector in $\mathbb{R}^3$. In particular, let $B_i = B(e_i)$.

It can be shown that $B(\xi)$ satisfies the equivariance property: $(R_g)_* B(\xi) = B(g^{-1}(\xi))$. (See [15], Chapter 2.) Again, viewing E as a Lie group $\mathscr{G}$, the fields B_i are left invariant vectorfields.

More generally, if $\mathscr{G}$ is the isometry group of a Riemannian manifold M, then $\mathscr{G}$ acts on the space $\mathscr{E}$ of orthonormal frames on the *left*. If $\mathscr{I}$ is an isometry, then $d\mathscr{I}(x) : M_x \to M_x$ is an isometry of the tangent space at x, and takes frame f to $d\mathscr{I}(x) \circ f$.

$$\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} M_x \xrightarrow{d\mathscr{I}(x)} M_x$$

This diagram shows how the isometries of M act (on the left) and the rotation group $SO(3)$ acts (on the right). The two actions commute.

Putting together the action of $SO(3)$ on the right and the action of the isometry group $\mathscr{G}$ on the left:

$$\mathscr{G} \times E \times SO(3) \longrightarrow E, \quad (\mathscr{I}, f, g) \longmapsto d\mathscr{I} \circ f \circ g$$

We see a nine-dimensional group $\mathscr{G} \times SO(3)$ acting on E, so there are lots of chances to reduce equations using symmetry.

The vector fields $A_1, A_2, A_3, B_1, B_2, B_3$ satisfy the Lie bracket formulas:

$$[A_i, A_j] = \varepsilon_{ijk} A_k \tag{13}$$
$$[A_i, B_j] = \varepsilon_{ijk} B_k \tag{14}$$
$$[B_i, B_j] = \varepsilon_{ijk} G A_k \tag{15}$$

where $\varepsilon_{ijk} = \pm 1$ depending on the sign of the permutation of $\{1,2,3\}$ and is 0 if two are equal. Formulas (13) and (14) hold on *any* 3 - manifold. Formula (15) holds for space forms of curvature G.

If V is a vectorfield on E, then the Hamiltonian $\mathscr{H}_V : T^*E \to \mathbb{R}$ is defined by $\mathscr{H}_V(p) = p(V)$, p any covector. This defines six *linear Hamiltonians* $\mathscr{A}_i, \mathscr{B}_i$ from A_i, B_i. The functions $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3$ are the generators of the algebra of left-invariant functions $\mathscr{L}\mathscr{G}$.

To compute Poisson brackets in this algebra, it suffices to compute brackets of the generators. For this we use the formula: $\{\mathscr{H}_V, \mathscr{H}_W\} = -\mathscr{H}_{[V,W]}$

In particular:

$$\{\mathscr{A}_i, \mathscr{A}_j\} = -\varepsilon_{ijk} \mathscr{A}_k$$
$$\{\mathscr{A}_i, \mathscr{B}_j\} = -\varepsilon_{ijk} \mathscr{B}_k$$
$$\{\mathscr{B}_i, \mathscr{B}_j\} = -\varepsilon_{ijk} G \mathscr{A}_k$$

An element of the algebra is a function $P(\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3)$ of six variables. 'Geometric' variational problems on curves give rise to left-invariant Hamiltonian systems.

The (generalized) Frenet equations for a framed curve are

$$\boxed{\begin{array}{lllll} \gamma'(t) = T & T' = & & k_2U & +k_3V \\ & U' = & -k_2T & & +k_1V \\ & V' = & -k_3T & -k_1U & \end{array}}$$

The usual Frenet equations correspond to $k_1 = \tau, k_2 = k, k_3 = 0$. If instead $k_1 = 0$, then we have the natural or inertial frame.

If $f(t) = (\gamma(t); T, U, V)$ is a curve in E, then the Frenet equations become:

$$\frac{df}{dt} = B_1(f) + k_1A_1(f) - k_3A_2(f) + k_2A_3(f) \tag{16}$$

This defines a control system: $k_i(t)$ are controls; given $f(0)$ we get a unique framed curve satisfying (16). Then we may seek controls satisfying the condition that the "cost"

$$c = \int \mathscr{L}(k_1, k_2, k_3)ds$$

is minimal.

Examples:

1. **Elastic curves** Using the standard Frenet frame ($k_3 = 0$), the cost functional is $\frac{1}{2}\int k_2^2 ds = \frac{1}{2}\int k^2 ds$. If instead we use the inertial frame ($k_1 = 0$), the cost functional is $\frac{1}{2}\int k_2^2 + k_3^2 ds = \frac{1}{2}\int k^2 ds$. This holds since $\nabla_T T = k_2U + k_3V = kN$.
2. **Kirkhoff rods** Using the general frame, the cost function is $\frac{1}{2}\int \alpha(k_3^2 + k_2^2) + \beta k_1^2 ds$
3. **τ-elastic rods** Consider the variational problem of minimizing total squared curvature for curves of fixed constant torsion $\tau = c$. Using the standard Frenet frame, the cost functional is $\frac{1}{2}\int k_2^2 ds = \frac{1}{2}\int k^2 ds$.

Given the control system and cost functional, one can apply the Pontrjagin Maximum Principle to produce a left-invariant Hamiltonian system on T^*E whose trajectories project to solutions of the optimal control problem, as follows:

1. Lift (16) to get a time-dependent Hamiltonian on T^*E (depending on control(s)) and subtract[4] the cost functional $\mathscr{L}$. This gives
$$\begin{aligned} \mathscr{H}(p; k_i) = \mathscr{B}_1(f) + k_1\mathscr{A}_1(f) - k_3\mathscr{A}_2(f) \\ + k_2\mathscr{A}_3(f) - \mathscr{L}(k_1, k_2, k_3) \end{aligned}$$
2. Maximize $\mathscr{H}$ with respect to choice of controls $k_i(t)$ (for each fixed t.) This is done by solving $\frac{\partial \mathscr{H}}{\partial k_i}$ for controls and eliminating them.
   The result is then a time-independent Hamiltonian.

We illustrate this with the example of the elastic rod.

$$\mathscr{H}(p; k_1, k_2, k_3) = \mathscr{B}_1 + k_1\mathscr{A}_1 - k_3\mathscr{A}_2 + k_2\mathscr{A}_3 - \frac{\alpha}{2}(k_2^2 + k_3^2) - \frac{\beta}{2}k_1^2$$

[4] This is a simplified description!

$$\frac{\partial \mathscr{H}}{\partial k_1} = 0 = \mathscr{A}_1 - \beta k_1$$
$$\frac{\partial \mathscr{H}}{\partial k_2} = 0 = \mathscr{A}_3 - \alpha k_2$$
$$\frac{\partial \mathscr{H}}{\partial k_3} = 0 = \mathscr{A}_2 - \alpha k_3$$

Eliminating the controls gives the Hamiltonian of the Kirchhoff elastic rod:

$$\mathscr{H} = \mathscr{B}_1 + \frac{\mathscr{A}_2^2 + \mathscr{A}_3^2}{2\alpha} + \frac{\mathscr{A}_1^2}{2\beta}$$

Now we can establish the Liouville-Arnol'd Integrability of the (symmetric) Kirchhoff rod problem. An important tool for this is the existence of two Casimirs for the algebra $\mathscr{L}\mathscr{G}$.

The quadratic Hamiltonians

$$\mathscr{P} = \mathscr{A}_1\mathscr{B}_1 + \mathscr{A}_2\mathscr{B}_2 + \mathscr{A}_3\mathscr{B}_3$$
$$\mathscr{Q} = \mathscr{B}_1^2 + \mathscr{B}_2^2 + \mathscr{B}_3^2 + G(\mathscr{A}_1^2 + \mathscr{A}_2^2 + \mathscr{A}_3^2)$$

are in the center of $\mathscr{L}\mathscr{G}$. That is, $\{\mathscr{P},\mathscr{H}\} = 0$ and $\{\mathscr{Q},\mathscr{H}\} = 0$ for all $\mathscr{H}$ in $\mathscr{L}\mathscr{G}$. This can easily be verified: we only need check on the generators $\mathscr{A}_i$ and $\mathscr{B}_i$ because of the product rule.

Let $\mathscr{R}\mathscr{G}$ be the algebra of *right* - invariant Hamiltonians; it is generated by the lifts of right-invariant vector fields. If $\mathscr{H} \in \mathscr{L}\mathscr{G}$ and $\mathscr{K} \in \mathscr{R}\mathscr{G}$, then $\{\mathscr{H},\mathscr{K}\} = 0$. (This is because left and right actions commute, so the vector fields commute).

So if $\mathscr{H} \in \mathscr{L}\mathscr{G}$, then by choosing $\mathscr{R}_1$ and $\mathscr{R}_2$ to be linear right-invariant Hamiltonians with $\{\mathscr{R}_1,\mathscr{R}_2\} = 0$ [which can be done in any of the three space-forms] we have five independent Hamiltonians in involution:

$$\mathscr{H}, \mathscr{P}, \mathscr{Q}, \mathscr{R}_1, \text{ and } \mathscr{R}_2.$$

To prove a given $\mathscr{H}$ is integrable, we need one more integral.

In our example, one can verify that $\mathscr{H} = \mathscr{B}_1 + \frac{\mathscr{A}_2^2+\mathscr{A}_3^2}{2\alpha} + \frac{\mathscr{A}_1^2}{2\beta}$ commutes with $\mathscr{A}_1$. The other Hamiltonians also commute with it automatically. Thus the Kirchhoff elastic rod is Liouville integrable.

Next consider the elastic curve. The usual Frenet frame does not work so well here, because the optimal control problem is *singular.* The equations can not be solved to eliminate the controls. Instead, we use the inertial frame. The resulting Hamiltonian is

$$\mathscr{H} = \mathscr{B}_1 + \frac{\mathscr{A}_2^2 + \mathscr{A}_3^2}{2}$$

Again, this commutes with $\mathscr{A}_1$, so the Euler elastica is integrable.

For the τ-elastic curve, there is only one control, so we can use the Frenet frame and eliminate the control. The resulting Hamiltonian is

$$\mathscr{H} = \mathscr{B}_1 + c\mathscr{A}_1 + \frac{\mathscr{A}_3^2}{2}$$

In this example it can be checked that $\mathscr{C} = \mathscr{B}_3 - c\mathscr{A}_3$ is a constant of motion *when the curvature* $G = 1$ *and* $c = \pm 1$.

One more example worth noting is given by $\mathscr{L}(k,\tau) = k^2\tau$. This leads to a non-polynomial example:

$$\mathscr{H} = \mathscr{B}_1 + \mathscr{A}_3\sqrt{\mathscr{A}_1}$$

Let

$$\mathscr{C} = \mathscr{A}_1^2 + \mathscr{A}_2^2 + \mathscr{A}_3^2 - 4\mathscr{A}_1\sqrt{\mathscr{B}_3} - 4G\mathscr{A}_1.$$

Then one can check that $\{\mathscr{C},\mathscr{H}\} = 0$; so this defines an integrable system.

For further examples of integrable geometric variational problems, see [22]. One intriguing *non-example* is the functional $\mathscr{L} = k^2 + \tau^2$. This does not appear to yield an integrable system.

Another source of geometric variational problems is found by replacing the orthonormal frame bundle with some other frame bundle. An interesting example of this is the notion of *affine* and *subaffine elastica*. The basic idea here is to replace Euclidean space with an affine space. (Good references for this kind of geometry include [28] and [33].) Given a curve, one can define a preferred frame by the condition that the volume be 1 at each point. In two dimensions, this is just the condition that $\det(\gamma,\gamma') \equiv 1$. From this it follows that $\gamma'' + \kappa\gamma = 0$ for some function κ, which is the *affine curvature*. An affine elastica is a critical point of $\int \kappa^2 d\sigma$, where σ is the (affine) arclength parameter. This turns out to be integrable; the related notion of subaffine elastica in three dimensions is also integrable. For details, see ([10], [11],and [31])

It is beyond the scope of these lectures to discuss soliton theory for infinite-dimensional Hamiltonian systems. We will only mention here that the integrability of the elastic rod is related to the integrability of the partial differential equation for an evolving space curve $\gamma(s,t)$:

$$\frac{\partial\gamma}{\partial t} = \frac{\partial\gamma}{\partial s} \times \frac{\partial^2\gamma}{\partial s^2}$$

This is known as the Betchov - Da Rios equation, also called the localized induction equation (LIE). A completely integrable Hamiltonian system, (LIE) possesses infinitely many conserved quantities, all of them integrals involving the curvature and torsion of the curve. The first four such quantities are the arclength, the total torsion, the total squared curvature, and the quantity $\mathscr{L}(k,\tau) = k^2\tau$. Note that the elastic rod is a critical point of a linear combination of the first three of these, while the fourth is constant along such a rod. For further details, see, e.g., [9].

5. CURVE STRAIGHTENING

Suppose a thin springy wire is fashioned into a smooth closed loop by joining the ends together, and the wire is held in some fixed configuration in $\mathbb{R}^3$. Then in the

Bernoulli-Euler model the bending energy is proportional to the total squared curvature. If we now release the rod and it moves in such a way as to reduce its energy as efficiently as possible, it will want to follow the "negative gradient" of the energy. (This is not physically realistic, however, since it assumes Aristotelian rather than Newtonian dynamics; also, we will ignore the fact that the wire can not physically pass through itself.) This is the idea behind the *Curve Straightening Flow*, which we will now be considering.

In order to actually define such a flow, we need several ingredients. First, we need the space of allowable curves, within which this flow will be defined. Next we need a way of defining a gradient, namely a Hilbert Space structure. Finally, we need to know that the flow is actually defined; that is, that the partial differential equation governing the flow has short-time and long-time solutions.

Assuming such a flow exists, we could then hope to describe the critical points of the functional (that is, the elastic curves) as limit points of trajectories of the flow. In fact, it turns out that under suitable hypotheses such a gradient flow is very well-behaved. (See [29] for the difficult technical details.) This allows us to rigorously determine existence and stability or instability of solutions in a wide range of situations.

First we formulate the appropriate definition of our space of curves. Let M be a Riemannian manifold. Let

$$\Omega = \{\gamma : [0,1] \longrightarrow M \mid \|\gamma'\| \equiv \ell \neq 0, \gamma'' \in L^2\}$$

and let

$$\Lambda = \{\gamma \in \Omega \mid \gamma(0) = \gamma(1), \gamma'(0) = \gamma'(1)\}$$

Λ is a Hilbert manifold: If M is Euclidean space we can define the norm by, for instance,

$$\|\gamma\|_2^2 = \gamma(0)^2 + \gamma'(0)^2 + \int_0^1 \|\gamma''\|^2 ds$$

For a general Riemannian manifold, the definition is rather more complicated. See [21] for the technical details.

$\mathscr{F}^\lambda : \Lambda \longrightarrow \mathbb{R}$ is defined by

$$\mathscr{F}^\lambda(\gamma) = \frac{1}{2}\int_\gamma k^2 + \lambda \, ds$$

More generally, A critical point of $\mathscr{F}^0$ with constrained length is a critical point of $\mathscr{F}^\lambda$ for some λ (Lagrange multiplier). λ may be thought of as a *length penalty*. Thus if $\lambda > 0$ arbitrarily long curves will have high $\mathscr{F}^\lambda$ values. Very short curves have high $\int k^2 ds$. So when λ is positive, the functional is bounded from below.

$\mathscr{F}^\lambda$ is a smooth function on a Hilbert manifold, so it defines a flow via the negative gradient, called Curve-Straightening. The key analytic result ([20], [21]) is:

Theorem 9 *If $\lambda > 0$, then $\mathscr{F}^\lambda$ satisfies the Palais-Smale condition (C). Therefore, the trajectories of $-\nabla\mathscr{F}^\lambda$ converge to critical points (or at least have critical points as adherence points). Furthermore the minimax principle and Morse theory hold for $\mathscr{F}^\lambda$.*

The theorem applies for M compact, and with modification for space forms. In the latter case, one can use congruence classes of curves to prevent trajectories from "escaping to infinity". The theorem also applies to spaces of curves of a fixed length.

Example: In $\mathbb{R}^2$, the only closed elastic curves are coverings of circles and figure-eight curves. There is precisely one critical point for $\mathscr{F}^\lambda$ ($\lambda > 0$) of rotation index $n \neq 0$; curve-straightening takes any closed curve of rotation index n to the n - fold circle. This demonstrates the Whitney-Graustein theorem, which asserts that any two closed curves of the same rotation index are regularly homotopic. In rotation index 0, the n-fold coverings of the figure eight curve are all critical points. Note that in this case Morse theory predicts multiple critical points; the space of closed curves of rotation index zero has nontrivial homology.

There are no critical points for $\lambda = 0$, because dilation reduces total squared curvature. Curve-straightening cannot satisfy the Palais-Smale condition in this case, since dilation causes curves to reduce total squared curvature; thus they will expand to infinite length.

Anders Linner has shown ([24]) that the curve straightening flow does not preserve convexity and that embedded curves need not stay embedded (although ultimately they must approach circles). See also [25] and [26].

Note: Yingzhong Wen initiated the study of "L^2 - curve straightening" [34], which has some nice geometric features. This is not actually a gradient flow in the sense described above, but rather a fourth-order semilinear parabolic differential equation. See [8] for recent interesting numerical work on this flow.

Recall the classification theorem of elastic curves in $\mathbb{R}^3$: For each pair of relatively prime integers (m,n) with $m > 2n$ there is a unique closed elastic curve (up to congruence)which lies on an embedded torus of revolution and represents an (m,n) torus knot.

Using the Palais-Smale condition one can prove more:

Theorem 10 *Let p and q be a pair of relatively prime integers with $0 < p \leq q$. Let $\mathscr{G}$ be the group of rotations around the z-axis generated by rotation through angle $\theta = \frac{2\pi p}{p+q}$. Then there is a non-circular closed elastica $\gamma_{p+q,p}$ which is $\mathscr{G}$-symmetric and $\mathscr{G}$-regularly homotopic to the p-fold circular elastica. It is a minimax critical point of total squared curvature (and hence unstable). It is only planar when $p = q$, in which case it is the figure-eight elastica.*

5.1. The Hyperbolic Plane

In $\mathbb{H}^2$, the closed elastic curves are much more abundant than in the Euclidean plane. In particular, there are critical points for $\lambda = 0$; we call such curves free elastica(e).

If the curvature of the hyperbolic plane is G, then the circle C of radius $\frac{\sinh^{-1}(1)}{\sqrt{-G}}$ is a free elastica, called the 'equator' of $\mathbb{H}^2$. (The name is by analogy to the equator of the sphere, which is a free elastica by virtue of being a geodesic.)

Free elastic curves can be classified using the Killing field $J(s) = \frac{k^2}{2}T + k'N$. Although rotation fields, translation fields, and horocycle fields all arise as examples of J, only the rotation fields are compatible with closed solutions. The following results are proved in [17]:

Theorem 11 *Let γ be a free elastica in $\mathbb{H}^2$. Then either γ is C^m for some m, or γ is a member of the family of solutions $\{\sigma_{m,n}\}$ having the following description:*

if $m > 1$ and n are integers satisfying $\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}$ there is (up to congruence) a unique curve $\{\sigma_{m,n}\}$ which closes up in n periods of its curvature $k = k_0 \operatorname{cn}(\frac{k_0 s}{2}, p)$ while making m orbits about the fixed point q of the rotation field J.

Theorem 12 *Let γ be a regular closed curve in $\mathbb{H}^2$, the hyperbolic plane with curvature G. Then*

$$\mathscr{F}(\gamma) = \int_\gamma k^2 \, ds \geq 4\pi\sqrt{-G}$$

with equality precisely for the equator C.

Among closed curves of rotation index 1 in the hyperbolic plane the equator is the only critical point for $\mathscr{F}$, and it is a global minimizer. One cannot conclude, however, that curve-straightening takes any closed curve to the equator, however, since the Palais-Smale condition fails to hold. This is an intriguing, and so far unresolved, issue. Work of Steinberg ([32]) gives supporting evidence for convergence of the flow in this case; however, see [26] for contrary evidence.

It is also interesting to note that among curves of rotation number $m \geq 3$, the multiply-wrapped equator is actually unstable, and there is no minimizer for $\mathscr{F}$ in such regular homotopy class.

Theorem 12 has an important application to a higher-dimensional geometric problem, namely that of *Willmore tori of revolution* in $\mathbb{R}^3$.

For the *Chen-Willmore problem*, one considers immersions $\Psi : M^2 \longrightarrow \mathbb{R}^3$ and the total squared mean curvature functional

$$\mathscr{H}(\Psi) = \int\int_M H^2 dA$$

where H is mean curvature and dA is the area element. Willmore showed in 1965 ([35]) that $\int\int_M H^2 dA \geq 4\pi$ on any closed surface in $\mathbb{R}^3$, with equality only for the round sphere.

The Chen-Willmore conjecture is that $\mathscr{H}(\Psi) \geq 2\pi^2$ when M is a torus. Robert Bryant ([5]) and Ulrich Pinkall independently observed the following:

Theorem 13 *Let γ be a regular closed curve in the hyperbolic plane represented by the upper half plane above the $x-$axis. If Ψ is the torus obtained by revolving γ around the $x-$axis, then $\mathscr{H}(\Psi) = \frac{\pi}{2}\mathscr{F}(\gamma)$*

From the inequality we derive [18]:

Corollary 14 *The Willmore inequality holds for tori of revolution.*

There are other ways to relate elastic curves to Willmore manifolds (critical points for total squared mean curvature). Let γ be critical for $\mathscr{F}^1$ in $\mathbb{S}^2$. U. Pinkall observed ([30]) that the inverse image of γ under the Hopf map $\pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is a Willmore torus; stereographic projection gives a Willmore torus in $\mathbb{R}^3$. If we apply Theorem 4, this gives an infinite family of embedded Willmore surfaces in $\mathbb{R}^3$. Variations on this theme can be found in, e.g., [2] and [3].

This construction also extends to higher dimensions. In [4], the Hopf map from the five-sphere $\mathbb{S}^5$ to complex projective space $\mathbb{C}P^2$ is used to pull back elastic curves to Willmore surfaces. In this case, there is no integrability result for elastic curves. However, special solutions can be explicitly obtained: among the elastic curves are a countable family of *closed* curves whose curvature, torsion, and third curvature are all constant: helices in $\mathbb{C}P^2$. These pull back to a family of Willmore surfaces with constant mean curvature in $\mathbb{S}^5$.

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