Notes on Riemann sums and Riemann integrals

Fact. The following are equivalent

(i) $f \in \mathcal{R}[a,b]$, i.e. $\exists L \forall \epsilon > 0 \exists \delta > 0 \forall \hat{P} \|\hat{P}\| < \delta \Rightarrow |S(f,\hat{P}) - S(f,\hat{Q})| < \epsilon$.
(ii) $\forall \epsilon > 0 \exists \delta > 0 \forall \hat{P}, \hat{Q} \|\hat{P}\| < \delta, \|\hat{Q}\| < \delta \Rightarrow |S(f,\hat{P}) - L| < \epsilon$.
(iii) $\exists L \forall (\hat{P}_n) \|\hat{P}_n\| \to 0 \Rightarrow S(f,\hat{P}_n) \to L$.
(iv) $\forall (\hat{P}_n) \|\hat{P}_n\| \to 0 \Rightarrow (S(f,\hat{P}_n))$ is a Cauchy sequence.

Equivalence of (i) and (ii) is the content of the “Cauchy Criterion” 7.2.1 in the textbook. All equivalences are fully analogous to, and their proofs follow closely similar statements for $\lim_{x \to c} f(x)$.

Given bounded function $f : [a,b] \to \mathbb{R}$ and a partition of $[a,b]$ (not tagged yet) $\mathcal{P} = (x_0, x_1, \ldots, x_n)$ we define the lower und upper sums of $f$ with respect to $\mathcal{P}$ by

$$L(f, \mathcal{P}) := \sum_{j=1}^{n} m_j (x_j - x_{j-1}), \quad U(f, \mathcal{P}) := \sum_{j=1}^{n} M_j (x_j - x_{j-1}),$$

where $m_j := \inf \{f(x) : x_{j-1} \leq x \leq x_j\}$ and $M_j := \sup \{f(x) : x_{j-1} \leq x \leq x_j\}$.

We note that if $\mathcal{P}$ is any tagged partition with the same intervals as $\mathcal{P}$, then

$$L(f, \mathcal{P}) \leq S(f, \hat{P}) \leq U(f, \mathcal{P}). \quad (1)$$

Moreover $L(f, \mathcal{P}) = \inf S(f, \hat{P})$ and $U(f, \mathcal{P}) = \sup S(f, \hat{P})$, where the inf and sup are extended over all tagged partitions with the same intervals as $\mathcal{P}$.

Proposition. $f$ is Riemann integrable iff $\sup_{\mathcal{P}} L(f, \mathcal{P}) = \inf_{\mathcal{P}} U(f, \mathcal{P})$.

One sometimes refers to $L(f) := \sup_{\mathcal{P}} L(f, \mathcal{P})$ and as the lower integral of $f$ over $[a,b]$ and to $U(f) := \inf_{\mathcal{P}} U(f, \mathcal{P})$ as the upper integral. Thus a function is integrable iff its lower and upper integrrals coincide. We note that $L(f)$ and $U(f)$ are defined for any function $f$ and one always has $L(f) \leq U(f)$ (this follows from the inequality $L(f, \mathcal{P}) \leq L(f, \mathcal{P} \cup \mathcal{Q}) \leq U(f, \mathcal{P} \cup \mathcal{Q}) \leq U(f, \mathcal{Q})$ valid for any two partitions $\mathcal{P}, \mathcal{Q}$ of $[a,b]$; see the Lemma below.) While $L(f)$ and $U(f)$ do not have all the properties we would expect from the integral, for example linearity, they have some weaker properties which are sufficient for developing the theory.

Proof of Proposition. Proof of $\Leftarrow$. We show that for any $\epsilon > 0$ there exists $\delta > 0$ such that, for any tagged partition $\hat{P}$ with $\|\hat{P}\| < \delta$, one has $S(f, \hat{P}) > L(f) - \epsilon$. Together with the analogous inequality $S(f, \hat{P}) < U(f) + \epsilon$ (which can
be shown similarly to, or derived formally from the one we prove by considering \( S(-f, \mathcal{P}) \), this implies that whenever \(|\mathcal{P}\| < \delta\) and \( L = L(f) = U(f)\), then \(|S(f, \mathcal{P}) - L| < \epsilon\), whence integrability follows from the definition (the condition (i) above).

Denote \( M := \sup_{x \in [a,b]} |f(x)|\), fix \( \epsilon > 0 \) and let \( \mathcal{P}_0 = (y_0, y_1, \ldots, y_n) \) be any partition of \([a,b]\) such that \( L(f, \mathcal{P}_0) > L(f) - \epsilon/2\). Let \( \mathcal{P} = (x_0, x_1, \ldots, x_m) \) be any partition with \(|\mathcal{P}\| < \delta\) (\( \delta > 0 \)) to be chosen later) and \( \mathcal{P} \) any tagged partition obtained by fixing tags \((t_j)_{j=1}^{n}\) in \( \mathcal{P} \). Next, let \( \mathcal{P}_1 := \mathcal{P} \cup \mathcal{P}_0 \). Further, let \( \mathcal{P}_1 \) be a tagged partition whose underlying untagged partition is \( \mathcal{P}_1 \) and which has as tags all the tags \((t_j)_{j=1}^{m}\) of \( \mathcal{P} \) with the (at most \( n - 1 \)) additional tags for the intervals that do not contain an “old” tag chosen arbitrarily. By the Lemma below, \( L(f, \mathcal{P}_1) \geq L(f, \mathcal{P}_0) > L(f) - \epsilon/2\) while, by (1), \( S(f, \mathcal{P}_1) \geq L(f, \mathcal{P}_1)\), hence \( S(f, \mathcal{P}_1) > L(f) - \epsilon/2\). Since in fact we need a lower estimate for \( S(f, \mathcal{P})\), let us compare that quantity to \( S(f, \mathcal{P}_1)\). The terms of the former sum corresponding to intervals that were not changed by “throwing in” the extra partition points \((y_i)\) are identical to the corresponding terms of the latter sum. The remaining terms could have changed with \([x_{j-1}, x_j]\) split into two subintervals and \( f(t_j)(x_j - x_{j-1})\) replaced by two resulting terms. However, since there are at most \( n - 1 \) such split intervals, since \( x_j - x_{j-1} < \delta\), and since \( |f(x)| \leq M\), the total difference is less than \( 2M(n-1)\delta\). Accordingly,

\[
S(f, \mathcal{P}) > S(f, \mathcal{P}_1) - 2M(n-1)\delta > L(f) - \epsilon/2 - 2M(n-1)\delta;
\]

choosing \( \delta = \frac{\epsilon}{4M(n-1)}\) we get \( S(f, \mathcal{P}) > L(f) - \epsilon\), which is exactly that we wanted for the proof of the “only if” part.

For future reference, we point out that we actually showed that

(a) \( \forall \epsilon > 0 \ \exists \delta > 0 \ \forall \mathcal{P} \ |\mathcal{P}\| < \delta \Rightarrow L(f, \mathcal{P}) > L(f) - \epsilon \).

As earlier, this is equivalent to the sequential statement

(b) \( \forall (\mathcal{P}_n) \ |\mathcal{P}_n\| \to 0 \Rightarrow L(f, \mathcal{P}_n) \to L(f)\)

and we have similar statements for \( U(f, \mathcal{P}_n)\) and \( U(f)\).

**Proof of \( \Rightarrow \)** By the remark above, if \( |\mathcal{P}_n\| \to 0\), then \( L(f, \mathcal{P}_n) \to L(f)\) and \( U(f, \mathcal{P}_n) \to U(f)\). We now use our prior observation that \( L(f, \mathcal{P}) = \inf S(f, \mathcal{P})\) where \( \inf \) is extended over all tagged partitions with the same intervals as \( \mathcal{P} \) – to find, for each \( n\), a tagged partition \( \mathcal{P}_n'\) such that \( L(f, \mathcal{P}_n) \leq S(f, \mathcal{P}_n') < L(f, \mathcal{P}_n) + 1/n\). By the squeeze theorem for sequences, \( S(f, \mathcal{P}_n') \to L(f)\). Similarly we find tagged partitions \( \mathcal{P}_n''\) such that \( S(f, \mathcal{P}_n'') \to L(f)\). By construction, \( |\mathcal{P}_n''| = |\mathcal{P}_n'| = |\mathcal{P}_n| \to 0\) and so if \( L(f) \neq U(f)\), then the condition (iii) above shows that \( f\) is not Riemann integrable.

QED
We used above the following (proof left as an exercise)

**Lemma.** Let $P_0 = (y_0, y_1, \ldots, y_n)$ and $P_1 = (z_0, z_1, \ldots, z_N)$ be partitions of $[a, b]$ such that $P_1$ is a refinement of $P_1$ (i.e., $\{z_0, z_1, \ldots, z_N\} \supset \{y_0, y_1, \ldots, y_N\}$), then, for any $f$ (not necessarily integrable),

$$L(f, P_0) \leq L(f, P_1) \leq U(f, P_1) \leq U(f, P_0).$$

Consequently, if $P$ and $P_0$ are any partitions of $[a, b]$, then

$$L(f, P_0) \leq L(f, P \cup P_0) \leq U(f, P \cup P_0) \leq U(f, P_0).$$

Another argument for the “only if” part of the Proposition (which, however, is not self-contained) is as follows: Let $\epsilon > 0$ and choose a partition $P = (x_0, x_1, \ldots, x_n)$ for which $L(f, P) > L(f) - \epsilon/2$. Define a step function $\alpha$ to be equal to $m_j = \inf \{f(x) : x_{j-1} \leq x \leq x_j\}$ on the interval $[x_{j-1}, x_j]$ (for definiteness, on $[x_{n-1}, b]$ for $j = n$). Then $\alpha \leq f$ on $[a, b]$ and $\int_a^b \alpha = L(f, P) > L(f) - \epsilon/2$.

Similarly we find a step function $\omega \geq f$ such that $\int_a^b \omega < U(f) + \epsilon/2$. If $L(f) = U(f) =: L$, then it follows that $\int_a^b \omega - \int_a^b \alpha < (L + \epsilon/2) - (L - \epsilon/2) = \epsilon$, showing that $f$ is Riemann integrable by 7.2.3, the squeeze theorem for integrals.

We next show properties of $L(\cdot)$ and $U(\cdot)$ which are substitutes for linearity.

**Lemma.** $L(f + g) \geq L(f) + L(g)$ and $U(f + g) \leq U(f) + U(g)$.

**Proof for $L(\cdot)$.** It is enough to show that for every $\epsilon > 0$ there is a partition $P$ such that $L(f + g, P) > L(f) + L(g) - \epsilon$. To this end, choose partition $P'$ and $P''$ such that $L(f, P') > L(f) - \epsilon/2$ and $L(g, P'') > L(g) - \epsilon/2$. Set $P := P' \cup P''$, then $P$ is a refinement of both $P'$ and $P''$ and so by the previous lemma $L(f, P) \geq L(f, P') > L(f) - \epsilon/2$ and $L(g, P) \geq L(g, P'') > L(g) - \epsilon/2$.

This implies $L(f, P) + L(g, P) > L(f) + L(g) - \epsilon$. We now claim that $L(f + g, P) \geq L(f, P) + L(g, P)$, which combined with the preceding formula yields $L(f + g, P) \geq L(f) + L(g) - \epsilon$, as required. By definition, we have

- $L(f, P) = \sum_{j=1}^n \omega_j (x_j - x_{j-1})$, where $\omega_j := \inf \{f(x) : x \in [x_{j-1}, x_j]\}$;
- $L(g, P) = \sum_{j=1}^n \omega_j (x_j - x_{j-1})$, where $\omega_j := \inf \{g(x) : x \in [x_{j-1}, x_j]\}$;
- $L(f + g, P) = \sum_{j=1}^n \omega_j (x_j - x_{j-1})$, where $\omega_j := \inf \{f(x) + g(x) : x \in [x_{j-1}, x_j]\}$.

and our claim follows from $\omega_j \leq \omega_j$ valid for all $j$. (This is just the general fact $\inf_{x \in A} f(x) + \inf_{x \in A} g(x) \leq \inf_{x \in A} (f(x) + g(x))$.)

**Corollary.** If $f, g$ are Riemann integrable on $[a, b]$, then so is $f + g$ and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

**Proof** $\int_a^b f + \int_a^b g = L(f) + L(g) \leq L(f + g) \leq U(f + g) \leq U(f) + U(g) = \int_a^b f + \int_a^b g$. 

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