Extension theories for categories
(preliminary report)

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This paper is a slight revision of a paper I wrote in 1980 and never submitted for publication. I would greatly appreciate any information about more recent work on this topic.

1 Introduction

Let $\mathcal{C}$ and $\mathcal{A}$ be categories and $P : \mathcal{C} \to \mathcal{A}$ a functor which is bijective on objects and surjective on arrows; then $\mathcal{C}$ is an *extension* of $\mathcal{A}$. This notion of extension is studied here from the point of view of classifying and synthesizing extensions, by generalizing methods used in studying extensions of groups and semigroups. (Functors which merge objects don't behave so much like homomorphisms and probably require intrinsically categorical methods to study them.) More specifically, in this paper I describe how to study certain extensions of small categories by a method which is essentially the Eilenberg-Mac Lane theory of group extensions in a more general setting.

In Section 2 I describe how to generalize certain well-known concepts in semigroup theory to small categories. In Section 3 I define the particular type of extension to be considered (extensions by a right semifunctor). Theorem 3.1 constructs such extensions by factor sets, generalizing Schreier's theory for groups. Theorem 3.2 says that split-extensions correspond to trivial factor sets. Theorems 3.3 and 3.4 show that (under a certain restriction) split extensions are group objects in a certain comma category and extensions are simple transitive actions by such group objects. This generalizes the perception about group extensions of Beck [67]. In Section 4 I show how extensions by right semifunctors arise by a natural process (Theorem 4.1) and make a rather weak connection between categories of posets

(Theorem 4.2). In Section 5 I define for abelian-valued right semifunctors a generalization of the cochain complex induced by the bar resolution and show (Theorem 5.1) that cohomology in dimension 2 classifies extensions. This suggests that the cohomology in all dimensions is the same (with a dimension shift) as the appropriate triple cohomology, and this is stated as Theorem 5.2.

Practically every result in this paper is a straightforward generalization of the work of Leech [73], [78] for monoids, with the following exceptions: (1) Some of the generalized semigroup theoretic concepts in Section 2, like Green's relations. (2) Theorem 4.2, which is a not quite so straightforward generalization of a result of Grillet for semigroups. (3) Theorems 3.3, 3.4 and 5.2, which are generalizations of some of my own results for semigroups. All this means that this paper is really an exposition of some of Leech's work, with that work somewhat reformulated as in Wells [78], and most important *put in its proper generality* —small categories instead of monoids.

I should point out that there are other papers using semigroup-theoretic techniques to study categories, notably Elkins and Zilber [76], Wells [79], Nico [83].

In this paper I write functors and functions on the right, sometimes as exponents for clarity. Composition is from right to left. A category $\mathcal{A}$ is identified with its collection of arrows. Everything works for small categories, and most of the constructions work for large categories (notable exception: the functor Σ in Section 4).

2 Green's relations and Leech's categories

In this section I generalize certain basic concepts of semigroup theory to categories. Let $\mathcal{A}$ be a category and $F : \mathcal{A} \to \mathit{Set}$ a functor. For present purposes there is no loss in generality in assuming that F is *separated*: for distinct objects b, c of $\mathcal{A}$, bF and cF are disjoint. F induces an equivalence relation *mod* F on the elements of the various sets bF : if $u \in bF$, $v \in cF$, then $u \equiv v\,(\mathit{mod} F)$ if there are arrows $f : b \to c$ and $g : c \to b$ for which $u.fF = v$ and $v.gF = u$. In case $\mathcal{A}$ is a group, F is a permutation representation and the blocks ($\mathit{mod}\ F$) are the orbits. F also induces a category $\mathcal{F}\Delta$ and functor $F\Gamma = \mathcal{F}\Delta \to \mathcal{A}$ by a well-known construction due to Grothendieck. The objects of $\mathcal{F}\Delta$ are the elements of the various sets bF for objects b of $\mathcal{A}$. The arrows have the form $(u, f) : u \to u.fF$, where

$f = b \to c$ in $\mathcal{A}$ and $u \in bF$. Composition is given by

$$(u, f \circ (v, g) = (u, f \circ g) \tag{2.1}$$

where $v = u.fF, g : c \to d$ in $\mathcal{A}$. The functor $F\Gamma$ is the projection $u \to b$ if $u \in bF$ (remember F is separated) and $(u, f) \to f$. $F\Gamma : F\Delta \to \mathcal{A}$ is a "discrete opfibration" of $\mathcal{A}$, and the construction Γ yields an equivalence of the category of set-valued functors from $\mathcal{A}$ with the category of discrete opfibrations over $\mathcal{A}$; see Gray [69, pp. 245–249] or Barr and Wells [90, Chapter 11].

The *twisted arrow category* of $\mathcal{A}$ denoted $A\mathbb{D}$ (following Leech [73]) is $A(-,-)\Delta$, in other words the discrete opfibration over $\mathcal{A}^{op} \times \mathcal{A}$ corresponding to the hom-functor $A(-,-) : \mathcal{A}^{op} \times \mathcal{A} \to$ *Set*. $A\mathbb{D}$ may be realized concretely as the category whose objects are the arrows of $\mathcal{A}$, with an arrow from $m : b \to c$ to $m' : a \to a$ triple (k, m, n) of arrows such that

$$\begin{array}{ccc} b & \xrightarrow{\;m\;} & c \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle n} \\ a & \xrightarrow[\;m'\;]{} & d \end{array} \tag{2.2}$$

commutes. Composition becomes

$$(k, m, n) \circ (k', m', n') = (k' \circ k, m, n \circ n'). \tag{2.3}$$

The arrows of the form $(1_b, m, n)$ for all objects b of $\mathcal{A}$ form a subcategory $\mathcal{A}\mathbb{R}$ which is a disjoint union of the comma categories $\mathcal{A}(b,-)\Delta = b \downarrow A$ (the latter is the notation in Mac Lane [71]). Similarly the arrows $(k, m, 1_c)$ form a subcategory $\mathcal{A}\mathbb{L}$ which is the disjoint union of the $\mathcal{A}(-,c)\Delta$ (which are opposites of comma categories). Let $\mathcal{A}_* : \mathcal{A} \to \mathrm{Set}$ be the *covariant global hom-functor* defined for $B \in |\mathcal{A}|$ by

$$\mathcal{A}_*(b) = \underset{a \in |\mathcal{A}|}{\cup} \mathcal{A}(a, b) \tag{2.4}$$

and for $f : b \to b'$ in $\mathcal{A}$,

$$\mathcal{A}_*(f) = (m \mapsto m \circ f) : A_*(b) \to A_*(b'). \tag{2.5}$$

then $A\mathbb{R}$ is a realization of $\mathcal{A}\Delta$, and moreover, $A\mathbb{L}$ is a realization of $\mathcal{A}_*\Delta$ for the analogously defined contravarient global hom-functor.

$\mathbb{D}, \mathbb{L}$ and $\mathbb{R}$ all become functors from *Cat* to *Cat* this way: if $F : \mathcal{A} \to \mathcal{B}$, then $F\mathbb{D} : \mathcal{A}\mathbb{D} \to B\mathbb{D}$ is defined on arrows by

$$(k, m, n)F\mathbb{D} = (kF, mF, nF), \tag{2.6}$$

and $\mathbb{R}$ and $\mathbb{L}$ are defined by restriction of $\mathbb{D}$.

$\mathcal{A}\mathbb{R}$ and $\mathcal{A}\mathbb{L}$ generate $\mathcal{A}\mathbb{D}$ in the sense made precise by the following two propositions.

Proposition 2.1 Let $k : a \to b$, $m : b \to c$, $n : c \to d$ in $\mathcal{A}$. Then

$$\begin{array}{ccc} m & \xrightarrow{(k,m,1_d)} & k\circ m \\ \downarrow{\scriptstyle (1_b,m,n)} & \searrow{\scriptstyle (k,m,n)} & \downarrow{\scriptstyle (1_a,k\circ m,n)} \\ m\circ n & \xrightarrow[(k,m\circ n,1_d)]{} & k\circ m\circ n \end{array} \tag{2.7}$$

commutes, and moreover the upper (resp. lower) triangle in (2.7) is the only factorization of (k, m, n) into an arrow of $\mathcal{A}\mathbb{L}$ (resp. $\mathcal{A}\mathbb{R}$) followed by an arrow of $\mathcal{A}\mathbb{L}$).

Proof. Straightforward. See Leech [73, 78].

Proposition 2.2. Let $\mathcal{K}$ be a category and $F_L : \mathcal{A}\mathbb{L} \to K$, $F_R : \mathcal{A}\mathbb{R} \to \mathcal{K}$ be functors. Then there is a unique functor $F : \mathcal{A}D \to K$ restricting to F_L and F_R if and only if for all objects a, b, c of $\mathcal{A}$ and arrows $k : a \to b$, $m : b \to c$, $n : c \to d$,

$$mF_L = mF_R, \tag{2.8}$$

and the diagram

$$\begin{array}{ccc} mF & \xrightarrow{(k,m,1_c)F_L} & (k\circ m)F \\ \downarrow{\scriptstyle (1_b,m,n)F_R} & & \downarrow{\scriptstyle (1_a,k\circ m,n)F_R} \\ (m\circ n)F & \xrightarrow[(k,m\circ n,1_d)F_L]{} & (k\circ m\circ n)F \end{array} \tag{2.9}$$

commutes. Here $uF = uF_L = uF_R$. Moreover, if $G : \mathcal{A}\mathbb{D} \to K$ is also a functor and $\sigma = \{\sigma_f\}$ is a family of arrows of $\mathcal{K}$ indexed by the arrows of $\mathcal{A}$, then σ is a natural transformation from F to G if and only if it is both a natural transformation from F_L to G_L and a natural transformation from F_R to G_R.

Proof. Given (2.8) and (2.9), the rest of the proof is straight-forward.

There is one more Leech functor. The category $\mathcal{A}\mathbb{H}$ has arrows of $\mathcal{A}$ as objects. An arrow $(k, m, n) : m \to p$ must have $k \circ m = m \circ n = p$. Thus m and p must lie in the same homset and k and n must be endomorphisms. Composition is given by

$$(k, m, n) \circ (k', p, n') = (k'k, m, nn') = m \to k' \circ k \circ m. \tag{2.10}$$

$\mathbb{H}$ is defined as a functor in the same way as $\mathbb{D}$. Observe that $\mathbb{H}$ is *not* a subfunctor of $\mathbb{D}$.

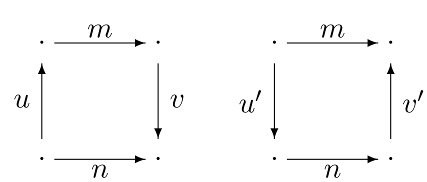
The *Green relations* on $\mathcal{A}$ are $\mathcal{J} = \mathit{mod}\ \mathcal{A}(-,-)$, $\mathcal{R} = \mathit{mod}\ \mathcal{A}_*$, $\mathcal{L} = \mathit{mod}\ \mathcal{A}^*$, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, and $\mathcal{D} = \mathcal{R} \cup \mathcal{L}$ (the equivalence relation generated by $\mathcal{R}$ and $\mathcal{L}$). These are all equivalence relations on the arrows of $\mathcal{A}$ and all specialize to the well-known Green relations on semigroups when $\mathcal{A}$ has only one object. The notation has been chosen to correspond to the standard notation for Green's relations. Most of the basic facts about Green's relations in semigroups generalize easily to categories, including Green's Lemma and the eggbox picture (see Howie [76] or Lallement [79]). I shall only state, mostly without proof, the results relevant here.

Proposition 2.3. $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} \subseteq \mathcal{J}$.

Proof: Easy. Here $\mathcal{L} \circ \mathcal{R}$ is the usual composition of relations: $x(\mathcal{L} \circ \mathcal{R})y$ means there is z for which $x\mathcal{L}z$ and $z\mathcal{R}y$.

It is well known that there are semigroups in which $\mathcal{D} \neq \mathcal{J}$. The example in Howie [76, II, Exercise 1] can easily be extended to a semigroup with unity element — hence a category — without affecting the fact that $\mathcal{D} \neq \mathcal{J}$. In terms of diagrams, if m, n are arrows of $\mathcal{A}$, then $m\mathcal{J}n$ if there are u, u', v, v' so that both squares in (2.11) commute:

(2.11)



In order that $m\mathcal{D}n$ there must be arrows u, u', v, v', so that all four squares in (2.11) and (2.12) commute:

$$(2.12)\qquad \begin{array}{ccc} \cdot & \xrightarrow{m} & \cdot \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle v} \\ \cdot & \xrightarrow[n]{} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xrightarrow{m} & \cdot \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle v'} \\ \cdot & \xrightarrow[n]{} & \cdot \end{array}$$

It is a nice exercise in diagram chasing to show that actually if both the squares in (2.11) and *one* of the squares in (2.12) commute, then the other square in (2.12) commutes.

I have been unable to decide whether $\mathcal{D} = \mathcal{J}$ in such well-known categories as groups, R-modules, topological spaces, and so on. However, $\mathcal{D} \neq \mathcal{J}$ in the category of semigroups since every semigroup with unity is the endomorphism semigroup of some semigroup, in particular the example with $\mathcal{D} \neq \mathcal{J}$ mentioned above. In *Set*, $f\mathcal{D}g \Leftrightarrow f\mathcal{J}g \Leftrightarrow$ the images of f and g have the same cardinality; $f\mathcal{L}g \Leftrightarrow f$ and g have the same image; and $f\mathcal{R}g \Leftrightarrow f$ and g induce the same equivalence relation on their common domain. Also, following Howie [76, Prop. II.1.5] one can show that if the endomorphism semigroup of every object of $\mathcal{A}$ is periodic, then $\mathcal{D} = \mathcal{J}$ for $\mathcal{A}$.

If Π is a group thought of as a one-object category, then $\Pi\mathbb{I}$, $\Pi\mathbb{R}$ and $\Pi\mathbb{D}$ are connected groupoids, that is, categories in which every arrow is an isomorphism and any two objects one isomorphic. Moreover, any two elements of Π are related by $\mathcal{H}$, hence $\mathcal{H} = \mathcal{R} = \mathcal{L} = \mathcal{D} = \mathcal{J}$.

A short exact sequence

$$(2.13)\qquad S : 1 \to K \overset{i}{\to} G \overset{\varphi}{\to} \Pi \to 1$$

with K Abelian determines a functor $F_L : \Pi\mathbb{L} \to \mathcal{A}b$ as follows. For $x \in \Pi$, $xFL = K$. Let $s : \Pi \to G$ be a set function splitting φ; I will call such s a *transversal* to φ. Define, for $x, u \in \Pi$ and $k \in K$:

$$(2.14)\qquad k.(v, x, l)F_L = v^s k v^{-s}.$$

Then $(v, x, l)F_L$ is an automorphism of K and F_L is a functor. If s is changed to another transversal, the resulting functor F_L is naturally isomorphic to the given one. If K is not Abelian, F_L can still be construed to be a functor by taking its codomain to be the quotient $[Grp]$ of the category of groups where homomorphisms $f, g : G \to H$ are identified if there is an inner automorphism i of H such that $f = g \circ i$.

This construction is just the well known action of Π on K (or abstract kernel, as in Mac Lane [63, IV Section 8], in the non-abelian case) made bizarre. However, formulating it this way is the clue toward developing a theory of extensions of categories. Going without inverses will force us to consider functors based on the $\mathbb{D}$-category instead of the $\mathbb{L}$-category, and to consider special kinds of congruences instead of kernels. In particular the functor corresponding to F_L won't be constant on objects. All this is worked out in detail in Section 3.

3 Extensions

In this section I define the basic notion of extension and show how to construct them by a generalization of the Schreier construction. Let $\mathcal{A}$ be a category. A mapping $F : \mathcal{A}\mathbb{D} \to \mathit{Grp}$ taking objects to objects and arrows to arrows is a *right semifunctor* if for $k : a \to b$, $m : b \to c$, $n : c \to d$ in $\mathcal{A}$, (3.1) through (3.3) are satisfied:

$$(1_b, m, 1_c)F = id_{mF}, \tag{3.1}$$

$$(1_a, k, m)F \circ (1_a, k \circ m, n)F = (1_a, k, m \circ n)F, \tag{3.2}$$

and

$$(1_b, m, n)F \circ (f, m \circ n, 1_d)F = (k, m, 1_c)F \circ (1_a, k \circ m, n)F = (k, m, n)F. \tag{3.3}$$

if F also satisfies

$$(m, n, 1_d)F \circ (k, m \circ n, 1_d)F = (k \circ m, n, 1_d)F, \tag{3.4}$$

then F is in fact a functor from $\mathcal{A}\mathbb{D}$ to Grp. I shall assume henceforth that any right semifunctor is separated.

Now let $\mathcal{C}$ and $\mathcal{A}$ be categories, $P : \mathcal{C} \to \mathcal{A}$ a functor which is the identity on objects and surjective on arrows, and $F : \mathcal{A}\mathbb{D} \to \mathit{Grp}$ a right semifunctor. Then $\mathcal{C}$ is a *Leech extension of* $\mathcal{A}$ by F if LE–1, LE–2 and LE–3 below are true. In this definition, and throughout the paper, composition in $\mathcal{A}$ and $\mathcal{C}$ is denoted by juxtaposition, whereas multiplication in groups in the image of F, as well as the action given by LE–1, is denoted by a fat dot: $\bullet$.

LE-1 (simple transitive local action). For each $m : b \to c$ in $\mathcal{A}$, the group in F acts simply transitively on the left on the fiber mP^{-1} of $\mathcal{C}$. The action will be denoted $(h, w) \to h \bullet w$ for $h \in mF$, $wP = m$.

LE-2 (composition on the right is equivariant). For all $k : a \to b$, $m : b \to c$ of $\mathcal{A}$, arrows v, w of $\mathcal{C}$ with $vP = k$, $wP = m$, and group element $g \in kF$,

$$\begin{array}{ccc} a & \xrightarrow{g \bullet v} & b \\ & {\scriptstyle g^{(1_a,k,m)F} \bullet (vw)} \searrow & \downarrow {\scriptstyle w} \\ & & c \end{array} \tag{3.5}$$

commutes. Note that kF acts on the fiber over km via the homomorphism $(1_a, k, m)F$, so the word "equivariant" is justified.

LE–3 (composition on the left is equivariant modulo 1-cochains). There is a function $\xi : \mathcal{C} \to \cup_{m \in \mathcal{A}} mF$ such that whenever $vP = k$ then $v\xi \in kF$, and for all k, m, v, w as in LE–2, and all $n \in mF$,

$$\begin{array}{ccc} a & \xrightarrow{v} & b \\ & {\scriptstyle h^{(k,m,1_c)F \circ \overline{v\xi}} \circ (vw)} \searrow & \downarrow {\scriptstyle h \bullet w} \\ & & c \end{array} \tag{3.6}$$

commutes, where $\overline{v\xi}$ is the inner automorphism

$$x \to (v\bar{\xi})^{(1_a,k,m)F} \bullet x \bullet (v\xi)^{-(1_a,k,m)F} \tag{3.7}$$

of $(km)F$.

If all the groups mF for all arrows in of $\mathcal{A}$ are Abelian, then F is a functor and the $\overline{v\xi}$ factor in LE–3 vanishes. In that case, $\mathcal{C}$ is an *Abelian Leech extension* of $\mathcal{A}$. In the general case, let $\bar{F}$ denote the canonical functor from *Grp* to [*Grp*] induced by F. Then $\bar{F}$ is a functor, the *abstract kernel* of the extension.

Two extensions $P : \mathcal{C} \to \mathcal{A}$ and $P' : \mathcal{C}' \to \mathcal{A}$ of $\mathcal{A}$ by the same functor F are *equivalent* if there is a category isomorphism $B : \mathcal{C} \to \mathcal{C}'$ over $\mathcal{A}$ such that

$$(g \bullet v)B = g \bullet (vB) \tag{3.8}$$

for all arrows $v = b \to c$ of $\mathcal{C}$ and all $g \in vPF$. The more general situation where $P' : \mathcal{C}' \to \mathcal{A}$ is an extension by a different right semifunctor F' which differs from F in the choice of ξ in LE–3 can be treated as in Leech [78,

p. 196] and Wells [78, p. 6 footnote]. Note that the abstract kernel is well-defined, i.e., $\bar{F} = \bar{F}'$.

The extension $P : \mathcal{C} \to \mathcal{A}$ is *split* if there is a functor $T : A \to \mathcal{C}$ which splits P. It is clear that an extension equivalent to a split extension is split.

The short exact sequence (2.13), together with a given transversal s (and not assuming K abelian) results in a right semifunctor $F : \Pi\mathbb{D} \to Grp$ defined by $xF = K$ for $x \in \Pi$ and

$$k(u, x, v)F = (u^s)x(u^s)^{-1} \qquad (k \in K, u, x, v \in \Pi) \tag{3.9}$$

— note the lack of dependence on v. Furthermore, G is a Leech extension of Π by F; the proof requires a lot of checking but is straightforward. The action of $xF = K$ on a coset $Kg(g \in G)$ is via multiplication. The functor F_L defined following (2.13) is F restricted to $\Pi\mathbb{I}$. The transversal S can sometimes be arranged so that Π acts on K via s; this corresponds precisely to the case that F is a functor.

Now I shall show how to synthesize all extensions of a category $\mathcal{A}$ by a right semifunctor; the syntheses requires "factor sets" which in Section 5 are interpreted as 2-cochains and which thereby give a classification of extensions by cohomology. This is all based on the work of Leech [73], [78] for monoids.

Let $\mathcal{A}$ be a category and $F : \mathcal{A} \to Grp$ a separated right semifunctor. Let $\mathcal{E} = \cup_{m \in \mathcal{A}} mF$; remember that $\mathcal{A}$ is identified with its collection of arrows. I shall make $\mathcal{E}$ the arrows of a category with the same objects as $\mathcal{A}$ so that $\mathcal{E}$ is an extension of $\mathcal{A}$ by F. The domain and codomain maps are given by: if $m : b \to c$ in $\mathcal{A}$ and $x \in mF$ then $x : b \to c$ in $\mathcal{E}$. For composition one needs factor sets: A function $\alpha\mathcal{A} \times \mathcal{A} \to \mathcal{E}$ is a *factor set* for F if for $k : a \to b$, $m : b \to c$, $n : c \to d$ and $z \in nF$,

$$(k, m)\alpha \in kmF \text{ (local definition)} \tag{3.10}$$

$$(k, 1_b)\alpha = (1_a, k)\alpha = 1_{kF} \text{ (normalized)} \tag{3.11}$$

$$\begin{aligned} &z^{(m,n,1_d)F\circ(k,mn,1_d)F} \\ &= (k, m)\alpha^{(1_a,km,n)F} \bullet a^{(km,n,1_d)F} \bullet (k, m)\alpha^{-(1_a,km,n)F}, \end{aligned} \tag{3.12}$$

and

$$(k, m)\alpha^{(1_a,km,n)F} \bullet (km, n)\alpha \tag{3.13}$$

$$= (m,n)\alpha^{(k,mn,1_d)F} \bullet (k,mn)\alpha \text{ (cocycle property).}$$

Observe that multiplication in (3.12) and (3.13) takes place in the group $kmnF$. With a given factor set α, the composition law in $\mathcal{E}$ is given by:

$$xy = x^{(1_a,k,m)F} \bullet y^{(k,m,1_c)F} \bullet (k,m)\alpha. \tag{3.14}$$

Finally, define $\Pi : \mathcal{E} \to \mathcal{A}$ by

$$x \bullet \Pi = k \text{ if } x \in kF. \tag{3.15}$$

Theorem 3.1 Let $\mathcal{A}$, F, $\mathcal{E}$ and Π be defined as above for a given factor set α. Then:

(a) With composition defined by (3.13), $\mathcal{E}$ is a category and Π is a functor.

(b) $\mathcal{E} \xrightarrow{\Pi} \mathcal{A}$ is a Leech extension of $\mathcal{A}$ by F with action by left group multiplication.

(c) Every extension of $\mathcal{A}$ by F is equivalent to an extension obtained as in (a) and (b) for some factor set α.

Proof: The proof of (a) and (b) is long but a straightforward generalization of the proof of Theorem 1 of Wells [78]. The proof of (c) requires the same technicalities as in the paper just mentioned; I will give the basic construction here. Let $\tau : \mathcal{A} \to \mathcal{E}$ be a function which is the identity on objects and on identity arrows, such that $\tau\sigma = \mathrm{id}_{\mathcal{A}}$, and such that for any $k : a \to b$, $m : b \to c$ in $\mathcal{A}_l$, $w : b \to c$ over m, and $h \in mF$:

$$k^\tau(h \bullet w) = h^{(k,m,1_c)F} \bullet (k^\tau w). \tag{3.16}$$

(It is possible to pick τ this way — see Wells [78, p. 6]). Such a τ is a *normalized transversal* for $\mathcal{E}' \xrightarrow{\sigma} \mathcal{A}$ and F. Define α implicitly by

$$k^\tau m^\tau = (k,m)\alpha \bullet (km)^\tau. \tag{3.17}$$

Then α is a factor set; let $\mathcal{E} \xrightarrow{\Pi} \mathcal{A}$ be constructed from α as in (a) and (b). The required equivalence $\beta : \mathcal{E} \to \mathcal{E}'$ of extensions is defined by

$$w = w^\beta \bullet m^\tau \tag{3.18}$$

for w in $\mathcal{E}$ such that $w\Pi = m$, $m : b \to c$ in $\mathcal{A}$.

Theorem 3.2. If $P : \mathcal{E} \to \mathcal{A}$ is a split extension of $\mathcal{A}$ by F then the splitting functor is a normalized transversal for the extension and with respect to that transversal the factor set α defined by (3.17) is trivial (i.e. $(k, m)\alpha = 1_{kmF}$ for all composable k, m). Conversely if α is trivial then the extension of $\mathcal{A}$ by F corresponding to α is split.

Proof: Routine verification. The splitting functor in the second sentence takes an arrow k of $\mathcal{A}$ to 1_{kF}.

I shall now show that under some restrictions a split Leech extension is a group object in a certain category and that any extension by the same right semifunctor is essentially a simple transitive group action by the group object. This requires some preliminary notions. The category *Gph* of graphs has as objects pairs (A, O) of sets with structure maps $d, c : A \to O$ and $i : O \to A$ such that

(3.19)
$$\begin{array}{ccc} O & \xrightarrow{\ i\ } & A \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow {\scriptstyle d} \\ & & O \end{array} \qquad\qquad \begin{array}{ccc} O & \xrightarrow{\ i\ } & A \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow {\scriptstyle c} \\ & & O \end{array}$$

commute. An arrow $F = (F_A, F_O) : (A, O) \to (A', O')$ consists of set functions $F_A : A \to A'$, $F_O : O \to O'$ which commute with structure maps:

(3.20)
$$\begin{array}{ccc} A & \xrightarrow{F_A} & A' \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ O & \xrightarrow[F_O]{} & O' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{F_A} & A' \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ O & \xrightarrow[F_O]{} & O' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{F_A} & A' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ O & \xrightarrow[F_O]{} & O' \end{array}$$

The *underlying graph functor* $U : \mathit{Cat} \to \mathit{Gph}$ takes a category $\mathcal{C}$ to the graph with $A =$ the set of arrows of $\mathcal{C}, O =$ the objects of $\mathcal{C}$, d, c the domain and codomain maps, and i the map taking an object to its identity arrow. U takes a functor to its arrow and object functions.

A right semifunctor $F : \mathcal{A}D \to \mathit{Grp}$ is *centralizing* if for all $k : a \to b$, $m : b \to c$ in $\mathcal{A}$, the image of $(1_a, k, m)F : kF \to kmF$ centralizes the image of $(k, m, 1_c)F : mF \to kmF$. (This condition is of course symmetric in k and m). A centralizing right semifunctor is actually a functor (use (3.12) and the remark concerning (3.4).)

Now let $P : \mathcal{E} \to \mathcal{A}$ be a split extension by a right semifunctor $F : \mathcal{A}D \to \mathit{Grp}$, with splitting functor $S : \mathcal{A} \to \mathcal{E}$. Define the set function $I : \mathcal{E} \to \mathcal{E}$ by requiring vI to be v^{-1} in the group kF, if $vP = k$. Also define a set function $M : \mathcal{E} \times {}_P\mathcal{E} \to \mathcal{E}$ by $(v,w)M = v \bullet W$ (multiplication in kF) if $vP = wP = k$; here $\mathcal{E} \times {}_P\mathcal{E}$ denotes the fiber product. Then S, I and M are all arrows in the comma category $Gph \downarrow \mathcal{A}U$ and the requisite diagrams forcing M to be associative, S the identity (observe that $\mathcal{A}U \overset{id}{\to} \mathcal{A}U$ is the terminal object on $Gph \downarrow \mathcal{A}U$) and I the inverse map are easily verified, making $PU : \mathcal{E}U \to \mathcal{A}U$ a group object in $Gph \downarrow \mathcal{A}U$.

On the other hand, suppose that $P : \mathcal{E} \to \mathcal{A}$ is a group object in $Cat \downarrow \mathcal{A}$ with group structure given by *functors* $S : \mathcal{A} \to \mathcal{E}$ (identity), $I : \mathcal{E} \to \mathcal{E}$ (inverse) and $M : \mathcal{E} \times_P \mathcal{E}$ (multiplication). It is easy to see that M induces a group multiplication on the each fiber of P. For each arrow $k : a \to b$ of $\mathcal{A}$ let kF be the fiber over k with this induced group structure. If $m : b \to c$, $n : c \to d$, let $(k,m,n)F : mF \to kmnF$ be defined by

$$w^{(k,m,n)F} = kS \circ w \circ nS. \tag{3.21}$$

Then $(k,m,n)F$ is a group homomorphism. It can be verified that $F : \mathcal{A}\mathbb{D} \to Grp$ is a functor.

I have now made the constructions for this theorem:

Theorem 3.3. If $P : \mathcal{E} \to \mathcal{A}$ is a split extension by a centralizing functor $F : \mathcal{A}D \to Grp$, then the maps S, I and M constructed above are functors, and P is a group object in $Cat \downarrow \mathcal{A}$. Conversely, if $P : \mathcal{E} \to \mathcal{A}$ is a group object in $Cat \downarrow \mathcal{A}$, then the functor $F : \mathcal{A}\mathbb{D} \to Grp$ constructed above is a centralizing functor and $P : \mathcal{E} \to \mathcal{A}$ is an extension of $\mathcal{A}$ by F split by S.

Proof. Straightforward, along the lines of the proof of Theorem 6 of Wells [78]. I do not know of an example of an $\mathcal{H}$-extension by a non-Abelian centralizing functor.

The following theorem generalizes Theorem 7 of Wells [78]:

Theorem 3.4. Let $P : \mathcal{E} \to \mathcal{A}$ be an extension by a centralizing functor $F : \mathcal{D}D \to Grp$. Let $Q : \mathcal{E}' \to A$ be the split extension by the same functor F. Then the group object corresponding to Q acts simply transitively on an object $P : \mathcal{E} \downarrow \mathcal{A}$ of $Cat \downarrow \mathcal{A}$ then P is an extension of $\mathcal{A}$ by F.

(Simple transitive action by a group object in an arbitrary category is defined by diagrams in the obvious way. See Wells [78, Section 4]).

Proof: Omitted.

4 $\mathcal{H}$-extensions

In this section I shall show that there are lots of extensions by right semifunctors by proving (Theorem 4.1) that every extension $P : \mathcal{C} \to \mathcal{A}$ whose corresponding congruence is included in $\mathcal{H}$ is an extension by a right semifunctor; such extensions are called $\mathcal{H}$-extensions. Since every category is an $\mathcal{H}$-extension of a category with no nontrivial $\mathcal{H}$-congruence (proof: factor out the maximal congruence contained in $\mathcal{H}$, just as in semigroups), this suggests that it is desireable to characterize categories with no nontrivial $\mathcal{H}$-congruence. Theorem 4.2 suggests a line of attack on this problem, but is not itself such a characterization.

Suppose that $P : \mathcal{C} \to \mathcal{A}$ is an $\mathcal{H}$-extension. I shall construct a right semifunctor $F : \mathcal{A}D \to \mathit{Grp}$; this construction is a generalization of the construction given in Section 3 for the group Π, and mimics the construction of Leech [76, Theorem 3.9]. Note that not every Leech extension is an $\mathcal{H}$-extension; for an example, see Leech [76, Section 5.16].

For each $m : b \to c$ in $\mathcal{A}$, pick $v : b \to c$ in $\mathcal{C}$ such that $vP = m$, and let $\mathcal{C}_m$ be the set of arrows $x : b \to b$ of $\mathcal{C}$ for which $(xv)P = vP = m$. Let E be the equivalence relation on $\mathcal{C}_m$ induced by requiring that xEx' if $xv = x'v$. Then composition is $\mathcal{C}$ induces a binary operation on the quotient set $\mathcal{C}_m/E$ which makes $\mathcal{C}_m/E$ a group. Let $mF = \mathcal{C}_m/E$. The group mF can be seen to be independent of the choice of v. I shall write $[x]_m$ to denote the element of mF induced by x. In general, x will have images $[x]_m$ in many different groups mF. The group mF acts simply transitively on the fiber mP^{-1} (the set of arrows of $\mathcal{C}$ over m); the action is induced by composition thus:

$$[x]_m \bullet v = xv. \tag{4.1}$$

Now assume s is a function from $\mathcal{A}$ to $\mathcal{C}$ which splits P in the sense that $m^s P = m$ for m an arrow of $\mathcal{A}$. As in the group case, s is a *transversal* of P. It follows that $m^s : b \to c$ in $\mathcal{C}$ if $m : b \to c$ in $\mathcal{A}$. I shall use s to define $(k, m, u)F$ for all triples (k, m, u) of arrows $k : a \to b$, $m : b \to c$, $n : c \to d$ of $\mathcal{A}$; $(k, m, u)F$ must be a group homomorphism $mF \to kmnF$. An element of mF is of the form $[x]_m$ for some $x : b \to b$ such that $xvP = m$ whenever $vP = m$. Let $y : a \to a$ in $\mathcal{C}$ be an arrow making this diagram commute:

$$\begin{array}{ccccccc}
a & \xrightarrow{y} & a & \xrightarrow{k^s} & b & \xrightarrow{m^s} & c \\
{\scriptstyle k^s}\downarrow & & & & & & \downarrow{\scriptstyle n^s} \\
b & \xrightarrow[x]{} & b & \xrightarrow[m^s]{} & c & \xrightarrow[n^s]{} & d
\end{array} \tag{4.2}$$

There is such an arrow because $xm^sP = m^sP = m$, hence $(k^sxm^sn^s)P = (k^sm^sn^s)P = kmn$, so that $k^sxm^sn^s\mathcal{H}k^sm^sn^s$. Then $y \in \mathcal{C}_{kmn}$ (this is proved by an easy argument using the fact that (4.2) commutes and any two arrows v, v' for which $vP = v'P = kmn$ are $\mathcal{H}$-related). Hence y defines an element $[y]_{kmn} \in (kmn)F$;

$$[x]_m(k,m,n)F = [y]_{kmn}. \tag{4.3}$$

It is straightforward using the simple transitivity of the group action mF on mP^{-1} to show that each $(k,m,n)F$ is a homomorphisms.

Theorem 4.1. Let $P : \mathcal{C} \to \mathcal{A}$ be an $\mathcal{H}$-extension with transversal s. Then F defined as above is a right semifunctor $F : \mathcal{A}\mathbb{D} \to \mathit{Grp}$ and $\mathcal{C}$ is a Leech extension of $\mathcal{A}$ by F.

Proof. The proof is essentially the same as in the semigroup case (Wells [78], Theorem 3) and will be omitted.

It is appropriate to call F the *Schützenberger functor* corresponding to P and s since the construction of the groups mF is essentially the same as the original construction of Schützenberger even though the construction here is much more general. The groups mF defined here are *left* Schützenberger groups. The right Schützenberger group for $m : b \to c$ can be defined using the set $\{y : c \to c | (wy)P = wP\}$, for $wP = m$. It is straight-forward using the fact that $P : \mathcal{C} \to \mathcal{A}$ is an $\mathcal{H}$-extension to show that the left and right Schützenberger groups are naturally isomorphic.

As an example of a naturally-occurring $\mathcal{H}$-extension, let Grp^o denote the category of groups and *surjective* homomorphisms, and for surjective group homomorphisms $f, g : G \to H$ let fEg if (as above) there is an inner automorphism i of H for which $g = f \circ i$. Let $[\mathit{Grp}^o]$ denote the corresponding quotient category. Then E is an $\mathcal{H}$-congruence and $\mathit{Grp}^o \to [\mathit{Grp}^o]$ is therefore an $\mathcal{H}$-extension.

If you try to generalize the preceding construction of all of Grp you will discover that there are two possibilities. One: Define E as above. Then E is an $\mathcal{R}$-congruence but unfortunately not an $\mathcal{H}$-congruence. By the way, E is not even a right coset congruence in the obvious generalization of that term — See Grillet [74]. Two: Define a relation E' by $fE'g \Leftrightarrow g = jf$ for some inner automorphism j of G (here $f, f' : G \to H$). There $E' \subset E$ and $E' \subset \mathcal{H}$. Unfortunately, E' is not a congruence! I do not know if there are any nontrivial $\mathcal{H}$-congruences on Grp.

Now I shall construct a functor which turns out to be an embedding when the domain category has no nontrivial $\mathcal{H}$-congruences.

Let $\mathcal{C}$ be a category and c an object of $\mathcal{C}$. Let cL denote the set of $\mathcal{L}$-equivalence classes of arrows with codomain c and cR the set of $\mathcal{R}$-equivalence classes of arrows with domain c. If m has codomain c, $[m]_L$ denotes the $\mathcal{L}$-equivalence class of m, and for n with domain $c[n]_R$ is analogously defined. The set cL is a poset, setting $[m]_L \leq [m']_L$ is there is f such that $m \circ f = m'$. (This does not depend on choice of representative). One makes cR a poset similarly (compose on the left). If $g : c \to d$, define gL by

$$[m]_L gL = [m \circ g]_L. \tag{4.4}$$

Then gL is an order-preserving map from cL to dL, and L is a functor from $\mathcal{C}$ to the category *Pos* of posets and order-preserving maps. Similarly if $f : b \to c$ set

$$[n]_R fR = [f \circ n]_R. \tag{4.5}$$

This makes $R : \mathcal{C} \to Pos^{op}$ a functor. In consequence there is a functor $\Sigma : \mathcal{C} \to Pos^{op} \times Pos$ determined by R and L and the universal property of products of categories.

Theorem 4.2. Σ is faithful if and only if the only congruence contained in $\mathcal{H}$ is trivial.

Proof. This theorem and proof generalizes Proposition 6.5 in Grillet [74]. Suppose α is a congruence on $\mathcal{C}$ contained in $\mathcal{H}$. Let $f : c \to d$, $g : c \to d$ and $f\alpha g$. If m has codomain c, then $mf\alpha mg$, so $mf\mathcal{H}mg$, so $mf\mathcal{L}mg$. Thus by (4.4) $fL = gL$. Analogously $fR = gR$, so $f\Sigma = g\Sigma$. Thus if Σ is faithful, the only congruence contained in $\mathcal{H}$ is the trivial one. Conversely, let σ be the congruence induced by $\Sigma : f\sigma g \Leftrightarrow f\Sigma = g\Sigma$. I shall complete the proof by showing that $\sigma \subset \mathcal{H}$. Suppose $f\sigma g$, with $f, g : c \to d$, then $fL = gL$ and $fR = gR$. Applying these functions to id_c and id_d respectively, you get that $f\mathcal{R}g$ and $f\mathcal{L}g$, hence $f\mathcal{H}g$.

5 Classification of extensions by cohomology

Beck [67] gives a general theory showing how in the presence of a triple simple transitive group actions by a group object in a category which are trivial in a certain sense relative to the triple are classified by the first triple cohomology group. If this is done in a comma category $\mathcal{C} \downarrow a$ the simple transitive group actions can be regarded as extensions of the object a by the group object. Group extensions in the usual sense are examples of such extensions, and so, via Theorem 3.4, are extensions of categories by right semifunctors. Eilenberg-Mac Lane (Mac Lane [63]) classify group extensions with Abelian kernel by the second cohomology group resulting from the bar resolution. The cohomology theory defined by Leech [73] generalizes the Eilenberg-Mac Lane cohomology; and Leech shows that in the monoid case his 2nd cohomology group classifies extensions by Abelian-valued functors. Here in Theorem 5.1 I will do for categories what Leech did for monoids. The Leech cohomology turns out to be that induced by the nerve of the category. Also, in Theorem 5.2 I state that this generalized Leech Eilenberg-MacLane cohomology coincides with a dimension shift with the triple cohomology relative to the triple generated by the functor $\mathcal{U} : \mathit{Cat} \to \mathit{Gph}$ of Section 3.

Let $F : \mathcal{A}\mathbb{D} \to \mathit{Grp}$ be an *Abelian*-valued functor. Let $\mathcal{W}$ be any category and $\sigma : \mathcal{W} \to \mathcal{A}$ a functor bijective on objects and surjective on arrows. Let $\mathcal{W}^{(O)}$ denote the set of objects of $\mathcal{W}$, $\mathcal{W}^{(1)}$ the set of arrows, and for $n > 1$, $\mathcal{W}^{(n)}$ the set of *composable chains of arrows* $(w_1, \ldots, w_n)$ of length n, meaning $\mathrm{cod} w_i = \mathrm{dom} w_{i+1}$ for $1 \leq i \leq n-1$. The set $C^n(\mathcal{W}, F)$ in *n-cochains with values in* $\mathcal{F}$ is the set of functions $\alpha : \mathcal{W}^{(n)} \to \cup_{m \in \mathcal{A}} mF$ satisfying

$$(w_1, \ldots, w_n)\alpha \in [(w_1 \ldots w_n)\sigma]F, \text{ and} \tag{5.1}$$

$$(w_1, \ldots, w_n)\alpha = O \text{ (if } w_i \text{ is identity for some } i). \tag{5.2}$$

A coboundary $\delta^n : C^n(\mathcal{W}, F) \to C^{n+1}(\mathcal{W}, F)$ is defined by

$$\begin{aligned}(w_1, \ldots, w_{n+1})\alpha\delta^n &= (w_2, \ldots, w_{n+1})\alpha^{(w_1^\sigma, N, 1_d)F} \\ &\quad + (-1)^{n+1}(w_1, \ldots, w_n)\alpha^{(1_a, M, w_{n+1}^\sigma)F} \\ &\quad + \sum_{k=1}^{n}(-1)^k (w_1, \ldots, w_{k-1}, w_k \circ w_{k+1}, w_{k+2}, \ldots, w_n)\alpha,\end{aligned} \tag{5.3}$$

where

$$M = w_1^\sigma \circ w_2^\sigma \ldots w_n^\sigma, \tag{5.4}$$

$$N = w_2^\sigma \circ \ldots \circ w_{n+1}^\sigma, \tag{5.5}$$

$a = \mathrm{dom} w_1^\sigma$, and $d = \mathrm{cod} w_{n+1}^\sigma$. In particular, the coboundary of a 1-cochain $\beta : \mathcal{W}^{(1)} \to F$ is given by

$$(5.6)\qquad (w_1, w_2)\beta\delta^1 = w_2\beta^{(w_1^\sigma, w_2^\sigma, 1_d)F} + w_1\beta^{(1_a, w_1^\sigma, w_2^\sigma)F} - (w_1 w_2)\beta,$$

and a 2-cochain α is a cocycle if it satisfies

$$(5.7)\qquad \begin{aligned} &(w_2, w_3)\alpha^{(w_1^\sigma, w_2^\sigma w_3^\sigma, 1_d)F} + (w_1, w_2 w_3)\alpha \\ &= (w_1, w_2)\alpha^{(1_a, w_1^\sigma w_2^\sigma, w_3^\sigma)F} + (w_1 w_2, w_3)\alpha, \end{aligned}$$

which becomes (3.13) if $\mathcal{W} = \mathcal{A}$, $\sigma = id_{\mathcal{A}}$. Let $H^n(\mathcal{W}, F)$ denote the nth cohomology group of the cochain complex $(C^*(\mathcal{W}, F), \delta)$.

The collections $\mathcal{W}^{(n)}$ are in fact the n-simplices of a certain simiplicial complex called the "nerve" of $\mathcal{W}$, and the coboundary map (5.3) is induced by the boundary map of the nerve; for details, see Duskin [74, Section 0.17].

Theorem 5.1. Let $F : \mathcal{A}\mathbb{D} \to \mathit{Grp}$ be an Abelian-valued functor. Then $H^2(\mathcal{A}, F)$ is in one-to-one correspondence with the set of equivalence classes of extensions of $\mathcal{A}$ by F.

Proof. A 2-cocycle in $C^2(\mathcal{A}, F)$ is a factor set: (3.10) follows from (5.1), (3.11) from (5.2), (3.12) is simply the fact that F is a functor, and (3.13) follows from (5.7) and the fact that α is a cocycle. Theorem 3.1 gives a map from $C^2(\mathcal{A}, F)$ *onto* the set of extensions of $\mathcal{A}$ by F. The Theorem then follows from the following Lemma.

𝕃⋗⋗∂ 5.2. Let α and α' be factor sets for F. Then the extension determined by α is equivalent to the extension determined by α' if and only if α is cohomologous to α'.

Proof: Let α determine $P : \mathcal{E} \to \mathcal{A}$ and α' determine $P' : \mathcal{E}' \to \mathcal{A}$. Let $B : \mathcal{E} \to E'$ be an equivalence of extensions. By Theorem 3.1(c) it may be assumed that $\mathcal{E} = \mathcal{E}' = \cup_{m \in \mathcal{A}} mF$, that multiplication in $\mathcal{E}$ is given by (3.13), and multiplication in $\mathcal{E}'$ by (3.14) with α replaced by α'. Because the groups are Abelian, I shall write + for $\bullet$ below. Let $k : a \to b$, $m : b \to c$ and $xP = xP' = k$, i.e. by our identification $x \in kF$, and $y \in mF$. Let xy denote the composite of x and y in E and $x * y$ the composite in E'. Then because B is a functor over $\mathcal{A}$,

$$(5.8)\qquad (xy)B = xB * yB,$$

and if $n : d \to e$ in $\mathcal{A}$,

$$(5.9)\qquad nB \in nF.$$

Define $\beta : \mathcal{A} \to \mathcal{E}$ by $n\beta = O_{nF}B$, by (5.9), $n\beta \in nF$, so is a 1-cochain of $\mathcal{A}$ in F. Now let $x = O_{kF}$, $y = O_{nF}$ in (5.8); apply (3.14) to both sides to get

$$\begin{aligned}(5.10) \qquad & (O_{kmF} + O_{kmF} + (k,m)\alpha)B \\ &= O_k B^{(1_a,k,m)F} + O_\mu B^{(k,m,1_c)F} + (k,m)\alpha'\end{aligned}$$

(remember the maps $(n_1, n_2, n_3)F$ are group homomorphisms so preserve the identity elements of the groups nF). Use commutativity, and (3.8) on the left side of (5.10) and apply the definition of β and you get

$$(5.11) \qquad (k,m)\alpha + (km)\beta = k\beta^{(1_a,k,m)F} + m\beta^{(k,m,1_c)F} + (k,m)\alpha',$$

so that α and α' differ by the coboundary of β, proving the Lemma.

The "underlying graph functor" $U : \mathit{Cat} \to \mathit{Gph}$ described in Section 3 is part of a triple in *Gph* with respect to which *Cat* is the category of triple algebras. The left adjoint of U is the functor $F : \mathit{Gph} \to \mathit{Cat}$ which takes a graph $A \rightrightarrows O$ to the following category $\mathcal{C}$: The objects of $\mathcal{C}$ are the elements of O. The arrows are composable chains of arrows of the graph modulo the equivalence relation $\sim$ generated by the following requirements:

$$(5.12) \qquad (f, i_y) \sim (i_x, f) \sim (f) \text{ for } f : x \to y, f \in A, x, y \in O$$

$$(5.13) \qquad (f) \sim (f') \Rightarrow (f,g) \sim (f',g) \text{ and } (h,f) \sim (h,f')$$

for all $f, f' = x \to y$, $g : y \to z$, $h : w \to x$.

Composition of arrows is induced by concatenation of chains. The multiplication $\mu : FUFU \to FU$ of the triple erases parenthesis just like the triple for groups in sets.

By a theorem of Beck [67] (stated as Theorem 4 in Wells [78]) if an adjoint pair $F : \mathcal{D} \to \mathcal{C}$, $U : \mathcal{C} \to \mathcal{D}$ is tripleable then "extensions" (suitably defined) by an Abelian group object y in $\mathcal{C}$ are classified by the first triple cohomology group with coefficients in y. These extensions turn out to be the same as extensions as defined in Section 3, so since Abelian group objects correspond to Abelian-valued functors as in Theorem 3.3, there must be a bijection between the first triple cohomology group and the 2nd cohomology group defined in Section 3. The two cohomologies turn out to be the same with a dimension shift in *all* dimensions; this is stated precisely as Theorem 5.2 below. The proof is the same as the proof of Theorem 8 of Wells [78]. (In connection with this, note that in Wells [78], the line above (6.1) on p. 32

should end this way: "H^nW is isomorphic to the nth cotriple cohomology group of W" — of course, the latter turns out to be $H^n_B(W,Y)_M$ in the application.

Theorem 5.2. Let $P : \mathcal{E} \to \mathcal{A}$ be an Abelian group object in $Cat \downarrow \mathcal{A}$, with corresponding functor $F : \mathcal{A}\mathbb{D} \to Grp$ as in Theorem 3.2. Let $\mathcal{W} \to \mathcal{A}$ be any object of $Cat \downarrow \mathcal{A}$. Then for all $n \geq 0$, the nth triple cohomology group of $\mathcal{W}$ with coefficients in P is naturally isomorphic to the $(n+1)$st cohomology group of $\mathcal{W}$ with coefficients in F as defined in Section 3.

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