On the Communication of Mathematical Reasoning*

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Abstract

This article discusses some methods of describing and referring to mathematical objects and of consistently and unambiguously signaling the logical structure of mathematical arguments.

Key words

Mathematical exposition, mathematical argument, formal reasoning, symbolic logic, proofs, terminology.

1 Introduction

This article discusses some methods of describing and referring to mathematical objects and of consistently and unambiguously signaling the logical structure of mathematical arguments that, in many texts, remain buried under a few unsystematic hints.

In section 2, we discuss the problems involved in referring to mathematical objects and properties. Section 3 discusses certain special aspects of definitions and makes some notational suggestions. Section 4 discusses certain individual words used to communicate the logical structure of mathematical arguments, and Section 5 suggests some strategies for writing proofs.

Many ideas of this paper are treated further in [28].

2 Referring to mathematical objects

2.1 Suggestive names

Mathematicians may name a type of object or property using a word that already exists in English. Such a word is usually chosen to suggest some aspect of the technical meaning. They also create words, usually from Latin or Greek roots, or name them after mathematicians. The Greek or Latin roots of a created name also may suggest the meaning of the word. We will use the phrase **suggestive name** to describe names that clearly suggest some aspect of the meaning to an educated person. In contrast, personal names and many created names (at least for readers unfamiliar with the roots used) are a kind of black box; the name gives no clue to the meaning.

In the next three subsections we consider three problems that can arise with names.

2.1.1 Cultural dependence The suggestiveness of a name will inevitably be **culture-dependent**. For example, the phrase “fixed point” is supposed to indicate that the point is thought of as standing still. The phrase is based on a use of the word “fix” that is uncommon in American colloquial English, in which “fix” most commonly means “repair”. One wonders whether a word such as “clockwise” will convey anything to students twenty years from now.

2.1.2 Cognitive dissonance The connotations of a word or phrase used to name a type of mathematical object sometimes create an expectation in the student that the object has properties different from the ones it actually has. This is a form of **cognitive dissonance**. The seminal work on cognitive dissonance is [5]. See also [3].

Example The use of the symbol $\subset$ in a context such as $A \subset B$ causes cognitive dissonance for many students. In texts in research mathematics the statement “$A \subset B$” commonly means that A is included as a subset in B, carrying no implication that A is different from B. However, students who are used to the difference between $m < n$ and $m \leq n$ often expect that $A \subset B$ should mean A is a *proper* subset of B and that one should express the idea that A is included in and possibly equal to B by saying $A \subseteq B$. Common usage in research thus fails to parallel the usage for inequalities; that is one type of cognitive dissonance. In recent years, usage has changed in high school and lower level college texts, so that $A \subset B$ is often expressly defined to mean that A is a proper subset of B.

Example In mathematical logic, the expression "$\exists r(r \in \mathbb{R}$ and $\pi r^2 < x)$" is an instance of what in some texts is called a **formula**. With the usual meanings of the symbols ($\mathbb{R}$ denotes the set of real numbers) the formula becomes a true sentence if x is instantiated at 3, and a false sentence if x is instantiated at -3. On the other hand, the expression πr^2 is an example of a **term**. When some real number is sustituted for r, this term evaluates to a number.

The word "formula" can cause cognitive dissonance. In colloquial usage, this word may refer to a complex expression for an object that allows one (with sufficient knowledge) to produce it. Thus, in chemistry the formula for hydrochloric acid is HCl. And in mathematics one might say that a formula for the area of a circle is πr^2. The way in which the word "formula" is used in these examples corresponds to what logicians call terms rather than to what they call formulas. Even students in graduate logic courses are sometimes confused by this.

On the other hand, other students, if asked what the formula for the area of a circle is, might respond "$A = \pi r^2$". For such students there may be no cognitive dissonance with the logician's use of the word "formula".

2.1.3 Overdependence on connotations In studying literature, the student may understand that the language is *given* to one complete with a vast baggage of cultural understanding. In consequence, literature brings into play complex meanings that depend on the reader's immersion in the ambient culture or on the reader's understanding of the culture in which the author worked.

Exposure to a liberal-arts tradition may therefore lead a student to expect, perhaps unconsciously, to extract the meaning of a word such as "group" in a text from the student's own experience with the word, from the context in which it appears, and from the cultural context. This student may not take the explicit definition of a word sufficiently seriously. Mathematicians and other scientists are used to inventing their own terminology. The definition of a word such as "group" in a mathematics text may require students to abandon most of their previous understanding of the word and start afresh, the exact opposite of what is expected in a literature course; and *students are seldom, if ever, told this.*

Mathematical authors are of course free to change the language. However, in doing so they are well advised to respect the givenness of the language and not to do such violence to it as would confuse or burden the user unreasonably. The meaning of a new technical word that is also a word in ordinary English should therefore be consonant as far as possible with the meaning and connotations of the word in everyday use. And the comments above about cultural differences and cognitive dissonance illustrate that suggestive notation may nevertheless have hidden traps.

We emphasize that the problem that students do not take the definition of a technical word seriously involves much more than being led astray by connotations. This is one of the main problems with teaching students who are just beginning to study abstract mathematics. The description of the way in which students cling tenaciously to their misconceptions about limits (only some of which are caused by connotations of the word "limit") in [25] is a good case study of this problem.

2.2 Type labeling

If it has been established on some early page of a text that S_3 denotes the symmetric group on three letters, then in referring to it many pages later in the book, an author is well advised to say, "Not every monoid with three elements can be embedded into the *group* S_3" instead of merely "Not every monoid with three elements can be embedded into S_3". This constitutes giving the **type** of the symbol S_3; i.e. it is of type "group". It may be desirable to use even more explicit typing, for example in the expression "the symmetric group S_3", where the phrase "symmetric group" gives the type of S_3.

We recommend declaring the type of nonstandard[1] symbols as illustrated above, whenever the symbols are used for the first time after an absence of several pages. Jeffrey Ullman, in a guest appearance in [12], flatly proposes *always* giving the type of a symbol. Russian authors of mathematics seem to do this a lot, although that may be because one cannot attach grammatical endings to symbols.

2.3 Associating fonts to types

Systematically using a particular font to typset symbols naming a particular type of mathematical entity (for instance, uppercase Roman for topological spaces, lowercase Greek for continuous maps) may give the reader a kind of subliminal support in keeping track of the different types of mathematical objects in the text. The authors in their research paper [1] carried out this idea explicitly. (This article is discussed further in section 2.5.)

[1]Of course, "standard" is a variable notion that depends on the subject matter and the expected educational level of the reader.

2.4 Suppression of parameters

One commonly refers to a structure, say a group $(G, \times)$, by the name of its underlying set. Since specifying a group requires specifying a set and a multiplication on it (with certain properties), referring to it by its underlying set alone is an example of **suppression of parameters**.

A more subtle example concerns the definition of continuity, which is commonly begun this way: "For every $\epsilon > 0$, there is a $\delta > 0$ for which..." Exhibiting the dependency of δ on ϵ explicitly could be accomplished by writing δ_ϵ or $\delta(\epsilon)$; however, the asserted δ is not uniquely determined by ϵ, and this notation suggests a function. One could always label the dependency explicitly: "For every $\epsilon > 0$, there is a $\delta > 0$ depending on ϵ for which..."

Functions are often referred to by the expressions which define them, as in the sentence, "The function x^2 is differentiable everywhere." This practice, too, constitutes suppression of parameters: To specify a function, one normally gives it a name, defines its domain and perhaps its codomain, and then gives a rule that allows one to evaluate the function at an element of the domain. The rule is commonly specified by an algebraic expression. To refer to "the function x^2" constitutes omitting all the parameters except the expression.

The styles of parameter suppression that we have mentioned are not going to go away soon. There are strong arguments in favor of continuing at least some of them. Students must be made aware of the phenomenon of parameter suppression when they have achieved the sophistication necessary to understand what it means. This may require quite a bit of work in some cases, for example when teaching the concept of function to beginning calculus students.

2.5 Mnemonic Symbols

The authors in their research paper [1] (already mentioned in 2.3) have experimented with using a system of symbolic names for mathematical objects and types of objects that will be called here the **Explicit Symbolic Style**. This style is distinguished by the following features:

1. Each name (particularly those involving functorial constructions) includes all the parameters (see 2.4), usually in square brackets.

2. An object or type is denoted by an abbreviation that includes sufficiently many letters to remind the reader of its meaning. (Dropping vowels and repeated consonants is a rough recipe.)

A style similar to the Explicit Symbolic Style is common in writing computer programs and the particular usage in the paper cited here is based on that of Mathematica.

Example : An article on groups that using the Explicit Symbolic Style could use $\mathsf{Ctr}[G]$ to denote the center of a group G. In the usual style of research papers and textbooks in group theory, one might denote the center of the group currently under discussion by C (thus omitting the name of the group and requiring the reader to remember that C stands for the center), but in the Explicit Symbolic Style one would refer to the center as $\mathsf{Ctr}[G]$ every time one mentions it.

Such a style would in many instances not be suitable for calculations involving the center of some specific group (particularly if made in private) because it is indeed easier to write C than Ctr. If one were trying to come up with a proof involving the center of G (typically sitting at a desk and writing down statements and formulas by hand) it would be inconvenient to have to write $\mathsf{Ctr}[G]$ every time. We would perhaps write C or C_G depending on the circumstances. Even if one were investigating the properties of the construction Ctr that depended on the group, one would be more likely in private to write $C(G)$ instead of $\mathsf{Ctr}[G]$.

We are not proposing that one write $\mathsf{Ctr}[G]$ in private; we propose only that this style be used in mathematical books and papers. Modern word processors and typesetting systems allow global changes and macros. This has made it easy to use symbols such as $\mathsf{Ctr}[G]$ in the final version of a book or paper. Thus the use of symbolic notation convenient for hand calculation need no longer jeopardize ease of understanding.

3 Definitions

In mathematical discourse, definitions are frequently signaled as such by the word "Definition" and by the fact that the defined word is put in italics or boldface. The latter device, however, is hardly universal. One could draw attention to this even more vividly as follows:

Definition: $n \mid m$ (read n **divides** m) $:\Leftrightarrow$ there exists $q \in \mathbb{Z}$ such that $m = qn$.

Definition: n is **even** $:\Leftrightarrow 2 \mid m$.
or alternatively,

Definition: n is said to be **even** $:\Leftrightarrow 2 \mid m$.

This format uses the Pascal colon to indicate that the equivalence holds by definition, not because it is true from previous remarks. The asymmetry of the symbol "$:\Leftrightarrow$" distinguishes what is being defined from the definition. This device is used by Noll [19] and Schäffer [21, page 3]. Following Noll and Schäffer, we suggest reading the symbol "$:\Leftrightarrow$" as "by definition if and only if".

The alternative form with "said to be" (which could also be "is called" in other situations) may be used as a convention that does not commit the author to the existence of mathematical objects. (Belief in the existence of mathematical objects is roughly speaking what is commonly called "Platonism".) This wording is intended to communicate the attitude that the statement "4 is even" is simply a way of talking, rather than a statement such as "Mars is red" that is about real things.

An analogous use of the colon with the equals sign (suggested by the use in Pascal) has become common; for example, given the radius r and circumference c of a circle, one may define $\pi := c/2r$.

Definitions are discussed further in the entry for "if" in the next section.

4 Signaling logical structure

In this section, we list a few common words and phrases that are typically used by mathematicians to express the logical structure of their arguments. Some of these have multiple meanings and others have meanings that differ from common English usage. The texts [4] and [10] are essentially attempts at providing a theoretical basis for the analogous problem of discovering the logical structure of discourse in ordinary English rather than in mathematical writing, and many of the problems discussed here are also described there.

a, an In mathematical writing, the indefinite article may be used with the name of a type of mathematical object to indicate an arbitrary object of that type. For example, the sentence "A positive integer is the sum of four squares" means the same as the formula

$$\forall n \exists a \exists b \exists c \exists d (n = a^2 + b^2 + c^2 + d^2)$$

where n, a, b, c, and d are all positive integers. The statement is true for every positive integer n without exception. Students asked to prove that *a* positive integer is the sum of four squares occasionally choose a particular integer (for example 12) and show that it is the sum of four squares, thinking they have done what the problem asked for. This usage of the indefinite article is deprecated by Gillman [7, page 7]. We recommend that mathematical authors write, "Every positive integer is the sum of four squares". See also the discussion under "any" below.

This usage occurs outside mathematics as well and is given a theoretical treatment in [10], section 3.7.4. In ordinary English sentences, such as

> A wolf takes a mate for life.

([10], page 294), the meaning is that the statement is true for a *typical* individual (typical wolf in this case). In mathematics, however, the statement is required to be true without exception. The idea that concepts are based on prototypes (typical examples) is one of the main points of [15] and [14].

any The word "any" in mathematical writing is one of many ways of expressing the universal quantifier. For example, the sentence, "Any positive integer can be written as a sum of four squares" means that *every* positive integer is the sum of four squares. However, just as mentioned in the discussion of "a, an", students sometimes interpret a sentence such as "Prove that any positive integer can be written as a sum of four squares" as an instruction to *pick* a positive integer and write it as a sum of four squares. Such misinterpretation does not seem to occur with "all" and "every". This discussion is based on Halmos' discussion in [23, p. 38].

if In definitions, it has been a convention to use "if" to mean "if and only if". It is probably better to *write* if and only if. Even that does not make it clear that this is a matter of definition rather than of logical equivalence. The colon convention described in Section 3 makes it completely clear that the statement is a definition.

Some might argue that the use of "if" in a *labeled* definition is a standard convention and is therefore unambiguous. This is a reason for teaching the students the convention; in this case, doing so is vital, because the convention is so common. But if a completely explicit way of pointing out a definition is possible, why not use it?

It is appropriate to make another point here about conventions. The colon convention for definitions has the advantage that every time it is used, it signals a definition. It is "context-free". The convention that "if" inside a labeled definition means "if and only if", on the other hand, is context-sensitive — the word "if" changes its meaning depending on the context in which it is used. This places more of a burden on the reader than a context-free convention does.

Of course, "if" is also used to signal an implication: A sentence such as "If $x > 4$, then $x > 2$" translates into the implication "$(x > 4) \Rightarrow (x > 2)$". In this connection, the word "implies" may improve clarity, since the resulting English sentence mirrors the structure of the symbolic notation more closely: "$x > 4$ implies that $x > 2$".

let This seems to have at least three different meanings.

In definitions: "Let $f(x) = x^2$", or "Let $S = \{1, 3, 5\}$." This appears to us to be used primarily to give local definitions: definitions that hold only in the current paragraph or subsection. (We thank one of the referees for this insight.)

To choose a witness: To pick an arbitrary object from a collection of objects known to be nonempty, or, to express it in logical terms, to choose a witness to an existential statement that is known to be true. For example:

> Given that G is a noncommutative group, let x and y be elements for which $xy \neq yx \ldots$

The following is a more explicit version of the same statement.

> Let the noncommutative group G be given. Since G is noncommutative, the collection $\{(x, y) \in G \times G \mid xy \neq yx\}$ is nonempty. Hence we may choose a member (x, y) of this set...

We prefer such an explicit version in texts for students inexperienced in mathematical proofs, because it makes a clear distinction between this use of "let" and the other two discussed here.

To prove a universal statement: To pick an arbitrary object from a collection with the purpose of proving a statement about all elements in the collection, using

Example: To prove that every even integer greater than 4 is the sum of two primes, one might begin: "Let n be an even integer greater than 4."[2] A way of making this usage more explicit for beginners would be to start the proof with, "Let the integer n be given. We assume that n is even and bigger than 4...", and to finish with a statement such as, "Since n was arbitrary, we may conclude...".

some The word "some" is used in mathematical discourse to indicate the existential quantifier. Therefore, the statement, "Some of the computers have sound cards", allows as a possibility that only one computer has a sound card, and it also allows as a possibility that all the computers have sound cards. *Neither of these interpretations reflect ordinary English usage*, in which the sentence quoted would lead one to expect that more than

[2] The authors are not sure how this proof would continue.

one computer has a sound card, and perhaps also that more than one does *not* have a sound card. In general, the passage from the quantifying English expressions in ordinary discourse to their interpretations as quantifiers in logical notation is fraught with difficulty. Some of the basic issues are discussed in [4, Chapter 3]; see also [10, 6].

the The indefinite and definite articles are used in a crucial way in mathematical writing, and it has become clear that students whose native language is not English often miss the significance of this usage. The problem can arise in exercises of this sort: "Let E be the set of even integers. Show that the sum of any two elements of E is even." Students have given answers such as this: "Let $E = \{2, 4, 6\}$. Then $2 + 4 = 6$, $2 + 6 = 8$ and $4 + 6 = 10$, and 6, 8 and 10 are all even."

This problem can be avoided by beginning, "Let E be the set of *all* even integers. . ."

when Used to mean "if", as in: "When a function has a derivative, it is necessarily continuous." Modern dictionaries [18] record this meaning of "when", but the original Oxford English Dictionary does not. The variant "whenever" is often used when otherwise there would be two "if"s in a row, as in: "A relation α is symmetric if whenever $x\alpha y$ then $y\alpha x$".

5 Consistent usage in proofs

The distinctions we have made in the preceding section concerning the words used to express logical structure show how little of that structure is actually signaled in most mathematical writing. Our suggestions concern the use of more precise words or phrases for certain specific purposes. It may be argued that *the same word or phrase should always be used to denote the same logical structure*. For example, always use "if", not "when" or "in the case that". Even the adoption of this practice along with the use of most suggestive possible word may fail to be clear enough; the student must be aware of what are after all *conventions* for associating English phrases with logical constructions.

5.1 The Telegraphic Style

The logical structure of a proof may be displayed explicitly by using notation from symbolic logic. The argument for doing this is that students should learn elementary logical notation just as they should learn algebraic notation, and, just as they should be able to follow an algebraic calculation embodied in a sequence of equations or inequalities, they should be able to follow a logical argument expressed in symbolic notation.

Most mathematicians writing about mathematical writing seem to recommend against excessive symbolism. See for example Halmos [23, page 38]; Schiffer [23, page 57]; Knuth, Larrabee and Roberts [12, page 1]; and Gilman [7, page 17].

However, some mathematicians, such as Gries [8] do advocate a free but judicious use of both logical and general mathematical symbols instead of words to ensure clarity. It seems reasonable to call such a style the **telegraphic style**. An example of such usage would be to replace the statement

A relation α on a set S is symmetric if and only if, for all $(x, y) \in S \times S$, if $x\alpha y$ then $y\alpha x$

by the telegraphic statement

$$(\alpha \subseteq S \times S \text{ is symmetric}) :\Leftrightarrow \forall(x, y) \in S{\times}S(x\alpha y \Rightarrow y\alpha x)$$

The telegraphic style may place a burden on some readers to the extent that they have to develop a facility in deciphering the symbolism. Its use may be justified when the concept being described is especially complex.

There is a feeling among some educators that logical symbols such as "$\forall$" and "$\Rightarrow$" *should* be as ubiquitous as π. It would be interesting to know if students who were taught symbolic notation at an early age could read and understand mathematics written in the telegraphic style as easily and rapidly as they could understand mathematics written in English prose (with nonlogical symbols but not logical symbols).

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