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ON THE LIMITATIONS OF SKETCHES

Dedicated to the memory of
Evelyn Nelson and Alan Day

MICHAEL BARR AND CHARLES WELLS

ABSTRACT. Call a category "sketchable" if it is the category of models in sets of some sketch. This paper explores the subtle boundary between sketchable and non-sketchable categories. We show that the category of small categories that have at least one initial object and functors that take an initial object to an initial object is sketchable. The same is true for weak initial objects, but is false for subinitial objects (that every object has at most one arrow to). Analogous results hold if we substitute finite limits for terminal object. We also show that the category of groups and center-preserving homomorphisms is not sketchable. We describe briefly how "higher-order" sketches can fill these gaps.

Introduction Sketches, as described for example in [Barr and Wells, 1985], can be used to describe many, but not all, kinds of mathematical structure. Recently Wells [1990] has described an extension of the notion to allow more powerful constructors than those given by limits and colimits to be used to describe structures. This raises the question of exactly what can be sketched with an ordinary sketch.

A theorem of Lair [1981] (rediscovered by Makkai and Paré [1990]) says that a category is sketchable if and only if it is accessible. This means that for some cardinal κ the category has colimits of all κ filtered diagrams and that every object is a κ filtered colimit of κ presentable objects. An object C of a category is κ presentable if the functor $\mathrm{Hom}(C, -)$ preserves the colimits of κ filtered diagrams. Since in practice one can usually decide quite easily whether a category is accessible, this gives a usable criterion for sketchability, without, unfortunately, giving any idea how to sketch certain theories.

Consider the category of categories with finite limits and functors that preserve them. We are not supposing canonical finite limits; the functors are merely required to take a finite limit diagram in the source to some finite limit diagram over the same base in the target. At first, it would seem that a theory to describe the set of all finite limit cones in a category would require a universal quantifier, and thus would not be sketchable. On the other hand, it is easy to see that the category of these categories with finite

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limits and functors that preserve them is $\aleph_0$ accessible and therefore from the theorem mentioned previously is sketchable. In this paper we actually exhibit a simple sketch for that category.

A similar argument works for the category of categories with weak finite limits (a cone is a weak limit if every other cone has at least one arrow to it). We indicate the minor change needed in the argument for the case of weak terminal objects.

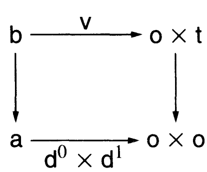
On the other hand, the related category of categories with sublimits (a cone is a sublimit if every other cone has at most one arrow to it) is not accessible and hence not sketchable. In this case, a universal quantifier or some higher order construct is needed. We show that a universal quantifier suffices. Among other things this shows that there is a class of first order sketches that has more expressive power than that of ordinary sketches.

One comment we should make concerns the use of the word "formal". A sketch is not a category. No diagram can commute nor can any object be the product of two others, a limit of a diagram or even a terminal object. When we say that a diagram formally commutes or that something is a formal limit or formal colimit, we mean that there is a diagram or cone or cocone in the sketch that will have the effect that the diagram becomes commutative or becomes a limit or colimit in any model.

1. The sketch for categories with finite limits

1.1. Categories with a terminal object We construct a sketch by beginning with a graph $\mathcal{G}$ with two ground sorts, called o (for objects) and a (for arrows). There will be other sorts, constructed as formal finite limits of these. We need a sort a_2 of formal composable pairs of arrows and a sort a_3 of formal composable triplets of arrows. As well, we need a sort 1, the formal terminal object of the sketch. We begin with the usual operations that give a category: $\mathsf{u}: \mathsf{o} \longrightarrow \mathsf{a}$ that assigns identity arrows to objects, d^0, $\mathsf{d}^1: \mathsf{a} \longrightarrow \mathsf{o}$ that assigns the domain and codomain to an arrow, and $\mathsf{c}: \mathsf{a} \times_{\mathsf{o}} \mathsf{a} \longrightarrow \mathsf{a}$ that composes composable pairs of arrows. We add the usual associative and left and right unit laws. This is the familiar sketch whose models in any category with finite limits are the category objects in that category.

To build a sketch for categories with a terminal object and functors that preserve them, we need two more sorts t (for terminal objects) and t' (which will also be for terminal objects), each formally subsorts of o. We let there be a sort b which is formally the subsort of a consisting of all arrows x for which $\mathsf{d}^1(x) \in \mathsf{t}$. We suppose an operation $\mathsf{v}: \mathsf{b} \longrightarrow \mathsf{o} \times \mathsf{t}$ such that the diagram



commutes. The arrows $\mathsf{b} \longrightarrow \mathsf{a}$ and $\mathsf{t} \longrightarrow \mathsf{o}$ are the formal inclusions. If we suppose that v is a formal isomorphism, then, in a model, each object has a unique arrow to each object of t, which means that the objects of t are terminal.

In order to force there actually to be a terminal object in a model, we suppose that the arrow $\mathsf{t} \longrightarrow \mathsf{1}$ is formally a regular epimorphism. We suppose of t' that it is defined formally as

$$\mathsf{t}' = \{x \in \mathsf{o} \mid \exists (y \in \mathsf{t}, f{:}\,x \longrightarrow y,\ g{:}\,y \longrightarrow x).f \circ g = uy \text{ and } g \circ f = ux\}.$$

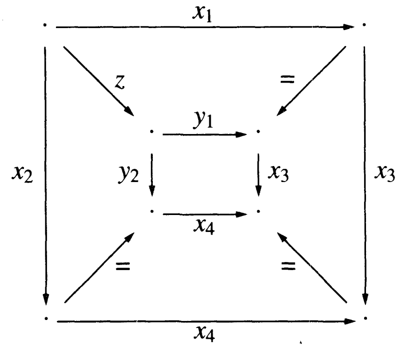
This is easily instantiated as the formal regular image in o of the first projection on the set of quadruples (x, y, f, g) that satisfy the equations described above. Formally, this condition makes t' into the set of all objects that are *isomorphic to* a terminal object. Finally, we map $\mathsf{t} \longrightarrow \mathsf{t}'$ by the operation that takes an $x \in \mathsf{t}$ to the image of (x, x, ux, ux) and then put in an inverse operation and equations to make this operation a formal isomorphism. The result of all this is that in a model M, $M\mathsf{t} = M\mathsf{t}'$ is non-empty and is actually the set of all terminal objects, so that a homomorphism between models is simply a functor that takes a terminal object in the one category to some terminal object in the second.

It is thus clear that the category of models in sets of this sketch is the category of small categories that have terminal objects and functors that preserve them.

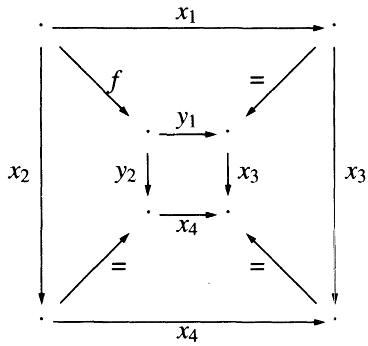
What about models in other categories? In a regular category in which supports split, the same claim is true. In general in a regular category, it is less clear what a model of this sketch is. A model is a category object for which the internal object of terminal objects has global support, but not necessarily a global section. It does not seem possible to build a sketch in which this object is guaranteed to have a global section without at the same time forcing that global section to be preserved by morphisms. One way of describing this is that this is a sketch for small categories in which terminal objects *exist internally*. We will ignore this point for the rest of the paper.

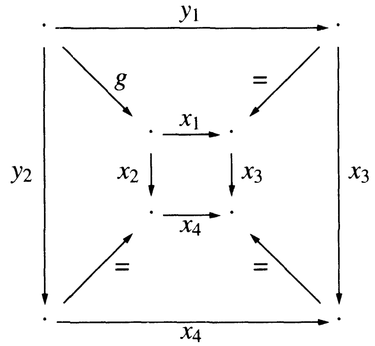
1.2. Categories with finite limits It is not hard to extend the above argument to finite limits or, indeed, the limits of any fixed set of shapes. Not surprisingly, it will require a sketch with limits of size greater than κ to sketch categories with limits of cardinality κ. We do the case of finite limits as an example, leaving other cases to the reader.

Begin with the sketch for categories with terminal objects. Add a sort s of all commutative squares. This is an equationally defined subsort of the sort of quadruples of arrows. Let p and p' be subsorts of s which are intended to be the set of pullback squares. There is a sort $\mathsf{r} \subseteq \mathsf{s} \times \mathsf{p}$ which formally consists of all 6-tuples $(x_1, x_2, x_3, x_4, y_1, y_2)$ of arrows such that (x_1, x_2, x_3, x_4) is in s and (y_1, y_2, x_3, x_4) is in p and another sort $\mathsf{q} \subseteq \mathsf{s} \times \mathsf{p}$ which consists of all 7-tuples $(x_1, x_2, x_3, x_4, y_1, y_2, z)$ of arrows such that $(x_1, x_2, x_3, x_4, y_1, y_2)$ is in r and $z{:}\, d^0(x_1) \longrightarrow d^0(y_1)$ is such that the following cube commutes:



To force the existence of all pullbacks, we need a sort h of pairs of arrows (y_3, y_4) with common codomain and a formal regular epi $\mathsf{p} \longrightarrow \mathsf{h}$. Finally, using the same technique as with terminal objects, we must force p' to consists of all pullback squares. Doing this is a bit more complicated than for terminal objects. We define a subsort w which is a subsort of $\mathsf{p} \times \mathsf{s} \times \mathsf{a} \times \mathsf{a}$ consisting of 8-tuples $(x_1, x_2, x_3, x_4, y_1, y_2, f, g)$ in which $(x_1, x_2, x_3, x_4) \in \mathsf{p}$, $(y_1, y_2, x_3, x_4) \in \mathsf{s}$ and f and g are morphisms for which the cubes





commute. In addition we need equations that make f and g formal inverse arrows to each other. Now p' must be required to be the formal regular image of the projection that takes such an 8-tuple onto (y_1, y_2, x_3, x_4). This forces p and p' to be the formal object of all pullback squares.

2. **Weak limits** A *weak limit* of a diagram is a cone over that diagram that satisfies the existence, but not the uniqueness, part of the definition. For example, an object of a category that every other object has at least one morphism to is called a weak terminal object. Just as with limits, one may ask for the category of categories that have weak limits over a set of diagram shapes and for functors that preserve them. As well, one might ask not for a chosen family of weak limits, but for simple existence and for functors that take all weak limits of those shapes to a weak limit. In fact, it is more appropriate in this case, since two weak limits of the same diagram will not generally be isomorphic, although they will have maps to each other.

The sketch for categories with weak terminal objects is just like the sketch for categories with a terminal object, with just one change: the operation v that was assumed to be an isomorphism is assumed only to be a formal regular epimorphism. If we modify the sketch for categories with finite limits by supposing only that both v and w are formal regular epimorphisms, then we get a sketch for categories with weak finite limits.

3. **Sublimits** A *sublimit* of a diagram is a cone with the property that every cone over that diagram has at most one arrow to that cone. For example, a subterminal object is an object that every object has at most arrow to. If the diagram has a limit, then a sublimit is any subobject of the limit. After seeing that both categories with limits and those with weak limits (of a certain shape or set of shapes) can be sketched, it comes as some surprise that the category of categories with subterminal objects and functors that preserve them cannot be sketched. We denote this category **SbTrm**.

We will show that **SbTrm** is not accessible, hence cannot be sketched. Since an accessible category must have all κ filtered colimits for some sufficiently large cardinal κ, it is sufficient to show that there are arbitrarily highly filtered diagrams in **SbTrm** that lack a colimit.

Fix a regular cardinal κ. Then the set of ordinals less than κ is κ filtered. We will construct a chain $\mathcal{C}_\lambda$, for $\lambda < \kappa$, of categories with subterminal objects and functors that preserve them that does not have a colimit in **SbTrm**. Each of the $\mathcal{C}_\lambda$ has the same set of objects: a set $\{A_\mu\}$, $\mu < \kappa$ and one more object A. In $\mathcal{C}_0$, there are two arrows $f_\mu, g_\mu: A_\mu \longrightarrow A$ for each $\mu < \kappa$. The category $\mathcal{C}_\lambda$ is the quotient of $\mathcal{C}_0$ gotten by identifying f_μ with g_μ for each $\mu < \lambda$. Let $\mathcal{C}$ be the category with the same set of objects and with just one arrow $A_\mu \longrightarrow A$ for each $\mu < \kappa$.

In each $\mathcal{C}_\lambda$, all the objects A_μ are subterminal. In addition A is subterminal in $\mathcal{C}$, but not in any of the $\mathcal{C}_\lambda$. There is a quotient mapping $\mathcal{C}_\mu \longrightarrow \mathcal{C}_\lambda$ for each $\mu < \lambda < \kappa$ and also $\mathcal{C}_\lambda \longrightarrow \mathcal{C}$ for $\lambda < \kappa$ and these clearly preserve the set of subterminals. It is clear that the colimit of the chain *in the category of small categories* exists and is $\mathcal{C}$. It cannot be the colimit in the category of categories with subterminal objects. For consider the category $\mathcal{C}'$ which is $\mathcal{C}$ together with another object B that has two arrows to A. The inclusion $\mathcal{C} \longrightarrow \mathcal{C}'$ does not preserve subterminal objects since A is subterminal in $\mathcal{C}$, but not in $\mathcal{C}'$. On the other hand the composite $\mathcal{C}_\lambda \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}'$ *does* preserve subterminal objects since A is not subterminal in $\mathcal{C}_\lambda$. But the only map $\mathcal{C} \longrightarrow \mathcal{C}'$ consistent with that chain of functors is the inclusion that we have just seen does not preserve subterminals. Thus $\mathcal{C}$ is not the colimit.

Now suppose that the chain has a colimit $\mathcal{D}$ in **SbTrm**. $\mathcal{D}$ must include objects we will call B_μ for $\mu < \kappa$ and B to be the images of the A_μ and A. They must all be distinct because there has to be a functor $\mathcal{D} \longrightarrow \mathcal{C}$ compatible with the quotient mappings. There are arrows $h_\mu, k_\mu: B_\mu \longrightarrow B$ to be the images of f_μ and g_μ. Since $f_\mu = g_\mu$ for $\mu > \lambda$, we must have $h_\mu = k_\mu$. On the other hand, there has to be a functor $\mathcal{D} \longrightarrow \mathcal{C}'$ so that B must not be subterminal in $\mathcal{D}$. Thus there must be at least one witness to the fact that B is not subterminal. This can either be another object B' that has at least two arrows to B or an arrow $k \neq h_\mu: B_\mu \longrightarrow B$ for some $\mu < \kappa$.

In the first case, let $\mathcal{D}'$ be the category constructed from $\mathcal{D}$ by adding an object $B'' \cong B'$. There are now two functors $\mathcal{D} \longrightarrow \mathcal{D}'$ that agree on all the $\mathcal{C}_\lambda$. The first is the inclusion and the second differs only in that it takes B' to B''. This takes care of the first case.

Suppose now that there is some cardinal μ such that in $\mathcal{D}$ there is a second arrow $l: A_\mu \longrightarrow A$. Let $\mathcal{D}'$ be the subcategory of $\mathcal{D}$ consisting of the objects and arrows of $\mathcal{C}$ together with l. This is a subcategory since there are no composable pairs (except those involving an identity arrow) in any of these categories. The inclusion $\mathcal{D}' \longrightarrow \mathcal{D}$ preserves subterminals and also includes the image of each $\mathcal{C}_\lambda \longrightarrow \mathcal{D}$. The resultant arrows $\mathcal{C}_\lambda \longrightarrow \mathcal{D}'$ also preserve subterminals. But it is a familiar fact of category theory that you cannot have a proper subobject of a colimit that factors all the arrows of the cocone, so it must be that $\mathcal{D}' = \mathcal{D}$. Now let $\mathcal{D}''$ be the category consisting of $\mathcal{D}$ together with a third arrow $l': A_\mu \longrightarrow A$. Then it is clear that there are two functors $\mathcal{D} \longrightarrow \mathcal{D}''$ that preserve subterminals and agree on every $\mathcal{C}_\lambda$, which contradicts the uniqueness of maps from a colimit. This shows that there cannot be a colimit in this category. Thus **SbTrm** is not accessible.

4. The category of groups with centers By the category of groups with centers, we mean the category **GpCen** whose objects are groups and whose morphisms are those that preserve the center. Using an argument similar to that for **SbTrm** we will show that the free group on two generators does not have a rank by showing that the square of the underlying functor—which is represented by that free group—does not preserve colimits of all κ filtered diagrams for any regular κ. It will be sufficient to show that the underlying functor does not preserve such colimits.

Let $X = \{x_\mu \mid \mu < \kappa\}$ be a set of variables in one-one correspondence with the cardinals less than κ. For $\lambda < \kappa$, let G_λ be the free group generated by X modulo the relations $x_\mu = x_\nu$ for all $\mu, \nu < \lambda$. The colimit G in the category of groups is free on one generator, which is commutative. If H is the free group on two generators x and y, the family of maps $G_\lambda \longrightarrow H$ that take all the generators to x does not extend to a map $G \longrightarrow H$ in the category. Thus the free group on one generator is not the colimit.

If there were a limit H' that commuted with the underlying functor, then there would have to be a map $H' \longrightarrow G$ that becomes an isomorphism when the underlying set functor is applied, which means it is already an isomorphism. This contradicts the fact that G is not the colimit.

5. A higher-order sketch for SbTrm and GpCen Although **SbTrm** and **GpCen** are not accessible, we give in this section constructions of higher order sketches of the type defined in [Wells, 1990] whose models in **Set** are those categories.

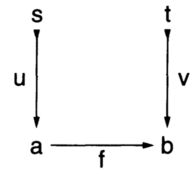
5.1. Higher order sketches A higher order sketch has a graph and may have diagrams, cones and cocones just like an ordinary sketch. In addition it may have other types of constructors. You specify the type of category the sketch is to have models in and then you can list as part of the data of the sketch a diagram D in the graph of the sketch that in a model must be a construction possible in such categories. Such a sketch has a theory and a generic model just like an ordinary sketch. These sketches are called "forms" in [Wells, 1990]. The exposition in [Power and Wells, 1990] gives a much more explicit description of higher order sketches, and generalizes them to allow the specification of 2-cells.

In the present case, the type of category that can have models of our sketch is a category $\mathcal{C}$ with finite limits and coequalizers of parallel pairs, for which, for each arrow $f: A \longrightarrow B$, the subobject functor Sub(f) has a right adjoint. The reason for asking for coequalizers is so we can specify that an arrow be a regular epimorphism.

A topos has these properties, and so does any locally cartesian closed category with coequalizers and a terminal object. For if $f: A \longrightarrow B$ in a locally cartesian closed category $\mathcal{C}$, then by definition the pullback functor $f^*: \mathcal{C}/B \longrightarrow \mathcal{C}/A$ has a right adjoint. Being a right adjoint, this functor must take monics to monics, so restricting it to Sub(B) gives the required right adjoint.

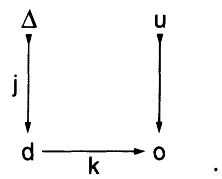
We will allow our higher order sketch to have a *universal quantifier constructor* ("UQ constructor"), a diagram shaped like this

(1)



In a model M, $M(\mathsf{u}): M(\mathsf{s}) \rightarrowtail M(\mathsf{a})$ must be monic and $M(\mathsf{v}): M(\mathsf{t}) \rightarrowtail M(\mathsf{b})$ must be the value of the right adjoint to $M(\mathsf{f})^*$ applied to $M(\mathsf{u})$. As a shorthand, we will say $\mathsf{v} = \forall_{\mathsf{f}}\mathsf{u}: \forall_{\mathsf{f}}\mathsf{s} \longrightarrow \mathsf{b}$. In **Set**, $M(\forall_{\mathsf{f}}\mathsf{s})$ will be the subset $\{b \in M(\mathsf{b}) \mid (\forall a \in M(\mathsf{a}))(M(\mathsf{f})(a) = b \Rightarrow a \in M(\mathsf{s})\}$.

5.2. The sketch for **SbTrm** We begin with the sketch for categories described in the first paragraph of 1.1. We need a node t to be the formal set of subterminal objects and a formal monic $\mathsf{e}: \mathsf{t} \longrightarrow \mathsf{o}$. Other nodes are: a node $\mathsf{a} \times \mathsf{a}$ and a cone forcing it to be the indicated formal product; a node Δ and arrow $\delta: \Delta \longrightarrow \mathsf{a} \times \mathsf{a}$ and a cone forcing it to be the formal diagonal; and a node d and arrow $\mathsf{i}: \mathsf{d} \longrightarrow \mathsf{a} \times \mathsf{a}$ which is to be the formal equalizer $[(f, g) \mid \mathsf{d}^0 f = \mathsf{d}^0 g$ and $\mathsf{d}^1 f = \mathsf{d}^1 g]$, together with an arrow $\mathsf{k}: \mathsf{d} \longrightarrow \mathsf{o}$ which formally takes a parallel pair to its codomain. Now adjoin an arrow $\mathsf{j}: \Delta \longrightarrow \mathsf{d}$. Since in a model, Δ must factor uniquely through d, this arrow will become a monic in a model. In order to force the existence of pullbacks, add a formal regular epi $\mathsf{t} \longrightarrow 1$. Finally, we need a UQ constructor



A model $\mathcal{C}$ of this sketch in **Set** must have

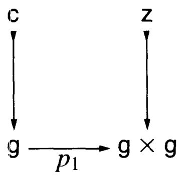
$$
(2) \qquad \begin{aligned}\mathcal{C}(\mathsf{t}) &= \{t \mid \text{for all } (f,g) \in \mathcal{C}(\mathsf{d}) \text{ with } d^1 f = t, (f,g) \in \mathcal{C}(\Delta)\} \\ &= \{t \mid \text{for all parallel pairs } (f,g) \text{ with codomain } t, f = g\}\end{aligned}
$$

Thus $\mathcal{C}(\mathsf{t})$ is the set of subterminal objects of $\mathcal{C}$. The formal regular epi $t \longrightarrow 1$ makes this set nonempty.

5.3. The sketch for **GpCen** The sketch for groups with centers is similar. What we want is to take the sketch for groups (using sorts $\mathsf{1}$, g, $\mathsf{g} \times \mathsf{g}$ and $\mathsf{g} \times \mathsf{g} \times \mathsf{g}$ and the familiar operations and identities necessary to define a group as well as the cones required to make the formal products) and add another type c equipped with a cone so that formally,

$$\mathsf{c} = \{(x,y) \in \mathsf{g} \times \mathsf{g} \mid xy = yx\}$$

and a UQ constructor



Then formally

$$\mathsf{z} = \{x \in \mathsf{g} \mid \forall y \in \mathsf{g}. xy = yx\}$$

and this is preserved by a group homomorphism if and only if the homomorphism preserves the center. Notice, by the way, that all group homomorphisms preserve the instantiation of c, for if f is a group homomorphism and $xy = yx$, then $f(x)f(y) = f(y)f(x)$. It is only the universal quantification that isn't preserved.

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