Affine invariant points *

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Abstract

We answer in the negative a question by Grünbaum who asked if there exists a finite basis of affine invariant points. We give a positive answer to another question by Grünbaum about the "size" of the set of all affine invariant points. Related, we show that the set of all convex bodies K, for which the set of affine invariant points is all of $\mathbb{R}^n$, is dense in the set of convex bodies. Crucial to establish these results, are new affine invariant points, not previously considered in the literature.

1 Introduction.

In the study of convex bodies, it is the affine geometry of these bodies that has become central in recent years. This is due to, on the one hand, rapid progress in core affine areas directly linked to the very structure of convex bodies, like the L_p Brunn Minkowski theory (e.g. [2, 5, 7, 8, 11], [18] -[26], [30, 35, 36, 38, 39]) and the theory of valuations [10], [12], [13], [15] - [17], [32, 33]. On the other hand, even questions that had been considered Euclidean in nature, turned out to be affine problems - among them the famous Busemann-Petty Problem (finally laid to rest in [3, 6, 41, 42]).

A key structural property of convex bodies is that of symmetry which is relevant in many problems. We only mention the still open Mahler conjecture about the the minimal volume product of polar reciprocal convex bodies (see e.g. [1, 14, 28] for partial results). The affine structure of convex bodies is closely related to the symmetry structure of the bodies. A systematic study of symmetry was initiated by Grünbaum in his seminal paper [9]. A crucial notion in his work is that of *affine invariant point*. It allows to analyze the symmetry situation. In a nutshell: the more affine invariant points, the fewer symmetries.

In this paper, we address several issues that were left open in Grünbaum's paper. We start out by laying the groundwork and we develop appropriate notions and tools. Then we answer some basic questions of Grünbaum. Firstly, Grünbaum asked whether the space of affine invariant points is infinite-dimensional. We give a positive answer in Theorem 1. Intuitively, this is clear since there should be a wealth of affine invariant points. Grünbaum though lists only very few, like the centroid and the Santaló point. Therefore, an important aspect in the proof is the construction of new affine invariant points. Secondly, Grünbaum asked whether there are convex bodies such that their affine

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invariant points cover the whole space $\mathbb{R}^n$. We show something even stronger, namely we show in Theorem 3 that such convex bodies are actually dense in the metric space of convex bodies.

Clearly, the affine invariant points of a convex body are invariant under the symmetries of this body. Grünbaum asked whether the set of all those points, that are invariant under all symmetries of the convex body, represent its affine invariant points. We answer this in the case that the set of affine invariant points has codimension 1. In fact, in Theorem 2 we construct explicitly a reflection with respect to a hyperplane that leaves the body invariant. To address these problems, we have to construct new affine invariant points. One of our tools to do that is the *convex floating body.*

Let $\mathcal{K}_n$ be the set of all convex bodies in $\mathbb{R}^n$ (i.e., compact convex subsets of $\mathbb{R}^n$ with nonempty interior). Then (see Section 2 for the precise definition) a map $p : \mathcal{K}_n \to \mathbb{R}^n$ is called an affine invariant point, if p is continuous and if for every nonsingular affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ one has,

$$p(T(K)) = T(p(K)).$$

An important example of an affine invariant point is the centroid g. More examples will be given throughout the paper. Let $\mathfrak{P}_n$ be the set of affine invariant points on $\mathcal{K}_n$,

$$\mathfrak{P}_n = \{p : \mathcal{K}_n \to \mathbb{R}^n \big| \ p \ \text{ is affine invariant}\}.$$

Observe that $\mathfrak{P}_n$ is an affine subspace of $C(\mathcal{K}_n, \mathbb{R}^n)$, the continuous functions on $\mathcal{K}_n$ with values in $\mathbb{R}^n$. We denote by $V\mathfrak{P}_n$ the subspace parallel to $\mathfrak{P}_n$. Thus, with the centroid g,

$$V\mathfrak{P}_n = \mathfrak{P}_n - g.$$

Grünbaum [9] posed the problem if there is a finite basis of affine invariant points, i.e. affine invariant points $p_i \in \mathfrak{P}_n$, $1 \leq i \leq l$, such that every $p \in \mathfrak{P}_n$ can be written as

$$p = \sum_{i=1}^{l} \alpha_i p_i, \quad \text{with } \alpha_i \in \mathbb{R} \quad \text{and} \quad \sum_{i=1}^{l} \alpha_i = 1.$$

We answer this question in the negative and prove:

Theorem 1. *$V\mathfrak{P}_n$ is infinite dimensional for all $n \geq 2$.*

In fact, we will see that, with a suitable norm, $V\mathfrak{P}_n$ is a Banach space. Hence, by Baire's theorem, a basis of $\mathfrak{P}_n$ is not even countable.

For a fixed body $K \in \mathcal{K}_n$, we let

$$\mathfrak{P}_n(K) = \{p(K) : p \in \mathfrak{P}_n\}.$$

Then Grünbaum conjectured [9] that for every $K \in \mathcal{K}_n$,

$$\mathfrak{P}_n(K) = \mathfrak{F}_n(K), \tag{1}$$

where $\mathfrak{F}_n(K) = \{x \in \mathbb{R}^n : Tx = x, \text{ for all affine } T \text{ with } TK = K\}$. We give a positive answer to this conjecture, when $\mathfrak{P}_n(K)$ is $(n-1)$-dimensional. Note also that if K has *enough symmetries*, in the sense that $\mathfrak{F}_n(K)$ is reduced to one point x_K, then $\mathfrak{P}_n(K) = \{x_K\}$.

Theorem 2. *Let $K \in \mathcal{K}_n$ be such that $\mathfrak{P}_n(K)$ is (n-1)-dimensional. Then*

$$\mathfrak{P}_n(K) = \mathfrak{F}_n(K).$$

Grünbaum [9] also asked, whether $\mathfrak{P}_n(K) = \mathbb{R}^n$, if $\mathfrak{F}_n(K) = \mathbb{R}^n$. A first step toward solving this problem, is to clarify if there is a convex body K such that $\mathfrak{P}_n(K) = \mathbb{R}^n$. Here, we answer this question in the affirmative and prove that the set of all K such that $\mathfrak{P}_n(K) = \mathbb{R}^n$, is dense in $\mathcal{K}_n$ and consequently the set of all K such that $\mathfrak{P}_n(K) = \mathfrak{F}_n(K)$, is dense in $\mathcal{K}_n$.

Theorem 3. *The set of all $K \in \mathcal{K}_n$ such that $\mathfrak{P}_n(K) = \mathbb{R}^n$ is open and dense in $(\mathcal{K}_n, d_H)$.*

Here, d_H is the Hausdorff metric on $\mathcal{K}_n$, defined as

$$d_H(K_1, K_2) = \min\{\lambda \geq 0 : K_1 \subseteq K_2 + \lambda B_2^n; K_2 \subseteq K_1 + \lambda B_2^n\}, \tag{2}$$

where B_2^n is the Euclidean unit ball centered at 0. More generally, $B_2^n(a, r)$, is the Euclidean ball centered at a with radius r. We shall use the following well known fact. Let $K_m, K \in \mathcal{K}_n$. Then $d_H(K_m, K) \to 0$ if and only if for some $\varepsilon_m \to 0$ one has

$$(1 - \varepsilon_m)\big(K - g(K)\big) \subset K_m - g(K_m) \subset (1 + \varepsilon_m)\big(K - g(K)\big) \text{ for every } m. \tag{3}$$

To establish Theorems 1 - 3, we need to introduce new examples of affine invariant points, that have not previously been considered in the literature.

2 Affine invariant points and sets: definition and properties.

Let $K \in \mathcal{K}_n$. Throughout the paper, $\mathrm{int}(K)$ will denote the interior, and ∂K the boundary of K. The n-dimensional volume of K is $\mathrm{vol}_n(K)$, or simply $|K|$. $K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \ \forall x \in K\}$ is the *polar body of* K. More generally, for x in $\mathbb{R}^n$, the *polar of K with respect to x* is $K^x = (K - x)^\circ$. Note that K^x is only bounded if $x \in \mathrm{int}(K)$. We will only consider such situations.

A map $p : \mathcal{K}_n \to \mathbb{R}^n$ is said to be continuous if it is continuous when $\mathcal{K}_n$ is equipped with the Hausdorff metric and $\mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$.

Grünbaum [9] gives the following definition of affine invariant points. Please note that formally we are considering maps, not points.

Definition 1. *A map $p : \mathcal{K}_n \to \mathbb{R}^n$ is called an affine invariant point, if p is continuous and if for every nonsingular affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ one has*

$$p(T(K)) = T(p(K)). \tag{4}$$

Let $\mathfrak{P}_n$ the set of affine invariant points in $\mathbb{R}^n$,

$$\mathfrak{P}_n = \{p : \mathcal{K}_n \to \mathbb{R}^n \big| \ p \ \ \textit{is affine invariant}\}, \tag{5}$$

and for a fixed body $K \in \mathcal{K}_n$, $\mathfrak{P}_n(K) = \{p(K) : p \in \mathfrak{P}_n\}$.

We say an affine invariant point $p \in \mathfrak{P}_n$ proper, if for all $K \in \mathcal{K}_n$, one has

$$p(K) \in \mathrm{int}(K).$$

Examples. Well known examples (see e.g. [9]) of proper affine invariant points of a convex body K in $\mathbb{R}^n$ are
(i) the *centroid*

$$g(K) = \frac{\int_K x dx}{|K|}; \tag{6}$$

(ii) the *Santaló point*, the unique point $s(K)$ for which the volume product $|K||K^x|$ attains its minimum;
(iii) the center $j(K)$ of the *ellipsoid of maximal volume* $\mathcal{J}(K)$ contained in K, or *John ellipsoid* of K;
(iv) the center $l(K)$ of the *ellipsoid of minimal volume* $\mathcal{L}(K)$ containing K, or *Löwner ellipsoid* of K.

Note that if $T(K) = K$ for some affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ and some $K \in \mathcal{K}_n$, then for every $p \in \mathfrak{P}_n$, one has $p(K) = p(T(K)) = T(p(K))$. It follows that if K is centrally symmetric or is a simplex, then $p(K) = g(K)$ for every $p \in \mathfrak{P}_n$, hence $\mathfrak{P}_n(K) = \{g(K)\}$.

The continuity property is an essential part of Definition 1, as, without it, pathological affine invariant points can be constructed. The next example illustrates this.

Example 1. *Let $\mathcal{P}_n$ be the set of all convex polytopes in $\mathcal{K}_n$ and define for $P \in \mathcal{P}_n$,*

$$p(P) = \frac{1}{m} \sum_{i=1}^{m} v_i(P),$$

where $v_1(P), \dots v_m(P)$ are the vertices of P. For $K \in \mathcal{K}_n \setminus \mathcal{P}_n$, let $p(K) = g(K)$, the centroid of K. Then $p : \mathcal{K}_n \to \mathbb{R}^n$ is affine invariant, but it is not continuous at any point.

Indeed, let $K \in \mathcal{K}_n$. We approximate K by a polytope P, and, in turn, approximate P by a polytope P_l by replacing one vertex v of P by sufficiently many vertices $v_1, \dots, v_l$ near v. When $l \to \infty$, $p(P_l) \to v \in \partial P$. P_l is near K, but $p(K) = g(K)$.

Next, we introduce the notion of affine invariant set mappings, or, in short, affine invariant sets. There, continuity of a map $A : \mathcal{K}_n \to \mathcal{K}_n$ is meant when $\mathcal{K}_n$ is equipped on both sides with the Hausdorff metric. Our Definition 2 of affine invariant sets differs from the one given by Grünbaum [9].

Definition 2. *A map $A : \mathcal{K}_n \to \mathcal{K}_n$ is called an affine invariant set mapping, if A is continuous and if for every nonsingular affine map T of $\mathbb{R}^n$, one has*

$$A(TK) = T(A(K)).$$

We then call $A(K)$, or simply the map A, an affine invariant set mappings. We denote by $\mathfrak{S}_n$ the set of affine invariant set mappings,

$$\mathfrak{S}_n = \{A : \mathcal{K}_n \to \mathcal{K}_n \big| A \ \text{ is affine invariant and continuous}\}. \tag{7}$$

We say that $A \in \mathfrak{S}_n$ is proper, if $A(K) \subset \mathrm{int}(K)$ for every $K \in \mathcal{K}_n$.

Known examples (see e.g., [9]) of affine invariant sets are the John ellipsoid and the Löwner ellipsoid. Further examples will be given all along this paper.

Remarks. (i) It is easy to see that if $\lambda \in \mathbb{R}$, $p, q \in \mathfrak{P}_n$ and $A \in \mathfrak{S}_n$, then $p \circ A \in \mathfrak{P}_n$ and $(1-\lambda)p + \lambda q \in \mathfrak{P}_n$. Thus, $\mathfrak{P}_n$ is an affine space and for every $K \in \mathcal{K}_n$, $\mathfrak{P}_n(K)$ is an affine subspace of $\mathbb{R}^n$. Moreover, for $A, B \in \mathfrak{S}_n$, the maps

$$K \to (A \circ B)(K), \quad (1-\lambda)A(K) + \lambda B(K) \quad \text{and} \quad \text{conv}[A, B](K) = \text{conv}[A(K), B(K)]$$

are affine invariant set mappings.

(ii) Properties (2) and (4) imply in particular that for every translation by a fixed vector x_0 and for every convex body $K \in \mathcal{K}_n$,

$$p(K + x_0) = p(K) + x_0, \text{ for every } \ p \in \mathfrak{P}_n \tag{8}$$

and

$$A(K + x_0) = A(K) + x_0, \text{ for every } \ A \in \mathfrak{S}_n \tag{9}$$

(iiii) Unless $p = q$, it is not possible to compare two different affine invariant points p and q via an inequality of the following type

$$\|p(K) - p(L)\| \geq c \, \|q(K) - q(L)\|, \tag{10}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and $c > 0$ a constant. Indeed, by (ii), $p(K - p(K)) = 0$ and $q(L - q(L)) = 0$. Therefore, if (10) would hold, then

$$\begin{aligned} 0 &= \|p(K - p(K)) - p(L - p(L))\| \geq c \, \|q(K - p(K)) - q(L - p(L))\| \\ &= c \, \|q(K) - p(K) - q(L) + p(L)\|. \end{aligned}$$

Choose now for L a symmetric convex body. Then $\|q(K) - p(K)\| = 0$, or $p(K) = q(K)$.

Remark (i) provides examples of non-proper affine invariant points: once there are two different affine invariant points, there are affine invariant points $p(K) \notin K$, i.e. non-proper affine invariant points. An explicit example is the convex body C_n constructed in [27], for which the centroid and the Santaló point differ.

The next results describe some properties of affine invariant points and sets.

Proposition 1. *Let $p, q \in \mathfrak{P}_n$ and suppose that p is proper. For $K \in \mathcal{K}_n$, define*

$$\phi_q(K) = \inf \left\{ t \geq 0 : q(K) - p(K) \in t \, (K - p(K)) \right\}.$$

Then $\phi_q : \mathcal{K}_n \to \mathbb{R}_+$ is continuous and

(i) there exist $c = c(q) > 0$ such that

$$q(K) - p(K) \in c \ \ (K - p(K)) \ \textit{ for every } K \in \mathcal{K}_n.$$

(ii) If moreover q is proper, then one can chose $c \in (0, 1)$ in (i).

Proof. Since $p(K) \in \text{int}(K)$, $\mathbb{R}^n = \cup_{t \geq 0} t(K - p(K))$. Therefore ϕ_q is well defined. Now we show that ϕ_q is continuous. Suppose that $K_m \to K$ in $(\mathcal{K}_n, d_H)$. By definition, we have

$$q(K_m) - p(K_m) \in \phi_q(K_m) \, (K_m - p(K_m)) \ \text{ for all } m.$$

By continuity of p and q it follows that

$$q(K) - p(K) \in \liminf_m \phi_q(K_m) \ (K - p(K)),$$

and thus

$$\phi_q(K) \le \liminf_m \phi_q(K_m).$$

Since p is proper, there exists $d > 0$, such that $B_2^n \subseteq d(K - p(K))$. Since K is bounded, there exists $D > 0$ such that $d(K - p(K)) \subseteq DB_2^n$. Let $\eta > 0$ and fix $\varepsilon = \eta/d > 0$. Since $K_m \to K$, and by continuity of p and q, there exists $m_0 > 0$ such that for every $m \ge m_0$,

$$K - p(K) \subseteq K_m - p(K_m) + \varepsilon B_2^n \subseteq K_m - p(K_m) + \eta(K - p(K))$$

and

$$q(K_m) - p(K_m) \in q(K) - p(K) + \varepsilon B_2^n \subseteq (\phi_q(K) + \eta)(K - p(K)).$$

Now we observe that, if two convex bodies A and B in $\mathbb{R}^n$ satisfy $A \subseteq B + tA$ for some $0 < t < 1$, then $A \subseteq B/(1-t)$. It then follows that for every $m \ge m_0$,

$$K - p(K) \subseteq \frac{1}{1-\eta}(K_m - p(K_m)).$$

Hence

$$q(K_m) - p(K_m) \in \frac{\phi_q(K) + \eta}{1-\eta} (K_m - p(K_m)),$$

and thus

$$\phi_q(K) \ge \limsup_m \phi_q(K_m),$$

and the continuity of ϕ_q is proved. Assertions (i) and (ii) follow from the continuity of ϕ_q. Indeed, by affine invariance, we may reduce the problem to the set $\{K \in \mathcal{K}_n : B_2^n \subseteq K \subseteq nB_2^n\}$, which is compact in $\mathcal{K}_n$. $\square$

Lemma 1. *Let $p, q \in \mathfrak{P}_n$ and suppose that p is proper. Then there exists a proper $r \in \mathfrak{P}_n$ such that q is an affine combination of p and r.*

Proof. By the preceding proposition, there is $c > 0$ such that $q(K) - p(K) \in c(K - p(K)$ for all $K \in \mathcal{K}_n$. Put $r = \frac{q-p}{2c} + p$. Then $r \in \mathfrak{P}_n$ and $q = 2cr + (1-2c)p$ is an affine combination of p and r. Since $p(K) \in \text{int}(K)$,

$$r(K) \in \frac{1}{2}(K + p(K)) \subseteq \text{int}(K), \quad \text{for all} \quad K \in \mathcal{K}_n. \qquad \square$$

Analogous results to Proposition 1 for affine invariant sets are also valid. We omit their proofs.

Proposition 2. *Let $A \in \mathfrak{S}_n$, $p, q \in \mathfrak{P}_n$ and suppose that p is proper. Then there exists a constant $c_1 > 0$ such that for every $K \in \mathcal{K}_n$,*

$$A(K) - q\big(K\big) \subseteq c_1 \ (K - p(K)).$$

If moreover A is proper and $p = q$, one can choose $c_1 < 1$.

Lemma 2. *Let $A \in \mathfrak{S}_n$ and $p \in \mathfrak{P}_n$ be proper. Then there exists $t > 0$ such that*

$$K \to t(A(K) - p(K)) + p(K) = tA(K) + (1-t)p(K)$$

is a proper affine invariant set mapping.

The next proposition gives a reverse inclusion for affine invariants sets. We need first another lemma, where, as in the proposition, g denotes the center of gravity.

Lemma 3. *For every $D, d > 0$ and $n \geq 1$, there exists $c > 0$ such that, whenever $K \in \mathcal{K}_n$ satisfies $K \subseteq DB_2^n$ and $|K| \geq d$, then $cB_2^n \subseteq K - g(K)$.*

Proof. Suppose that $K \in \mathcal{K}_n$ satisfies the two assumptions. Define

$$c_K = \sup\{c \geq 0 : \; cB_2^n \subseteq K - g(K)\}.$$

Then $c_K > 0$, and there exists $x \in \partial K$ such that $\|x - g(K)\| = c_K$. Since $K - g(K) \subseteq n\big(g(K) - K\big)$, the length of the chord of K passing through $g(K)$ and x is not bigger than $(n+1)c_K$. Let $u \in S^{n-1}$ be the direction of the segment $[g(K), x]$ and let $P_u K$ be the orthogonal projection of K onto $u^\perp$, the subspace orthogonal to u. Then,

$$d \leq |K| \leq \|(g + \mathbb{R}u) \cap K\| \; |P_u K| \leq (n+1)c_K \; D^{n-1}|B_2^{n-1}|.$$

The second inequality follows from a result by Spingarn [37]. Thus we get a strictly positive lower bound c for c_K which depends only on n, d and D. $\square$

Proposition 3. *Let A be an affine invariant set mapping. Then there exist $c > 0$ such that*

$$c\big(K - g(K)\big) \subseteq A(K) - g\big(A(K)\big), \quad \textit{for every } K \in \mathcal{K}_n.$$

Proof. We first prove that there exists $d > 0$ such that $|A(K)| \geq d|K|$ for every $K \in \mathcal{K}_n$. By affine invariance, it is enough to prove that

$$\inf_{\{K \in \mathcal{K}_n : B_2^n \subset K \subseteq nB_2^n\}} \frac{|A(K)|}{|K|} > 0.$$

Since $K \to \frac{|A(K)|}{|K|}$ is continuous and since $\{K \in \mathcal{K}_n : B_2^n \subseteq K \subset nB_2^n\}$ is compact in $\mathcal{K}_n$, this infimum is a minimum and it is strictly positive. By Proposition 2, applied with $q = g \circ A$, there exists $c > 0$ such that

$$A(K) - g\big(A(K)\big) \subseteq c\big(K - g(K)\big), \quad \text{for every } K \in \mathcal{K}_n.$$

Therefore,

$$A(K) - g\big(A(K)\big) \subseteq 2ncB_2^n, \quad \text{for every } K \in \mathcal{K}.$$

By Lemma 3, there is $c_0 > 0$ such that

$$c_0 B_2^n \subseteq A(K) - g\big(A(K)\big), \quad \text{for every } K \in \mathcal{K}.$$

Now, since $K - g(K) \subseteq 2nB_2^n$ for every $K \in \mathcal{K}_n$ with $B_2^n \subseteq K \subset nB_2^n$, we get the result for $K \in \mathcal{K}_n$ with $B_2^n \subseteq K \subset nB_2^n$, and thus for all $K \in \mathcal{K}_n$ by affine invariance. $\square$

3 Several questions by Grünbaum.

We now give the proof of Theorems 1 - 3. To do so, we first need to introduce new affine invariant points.

3.1 The convex floating body as an affine invariant set mapping.

Let $K \in \mathcal{K}_n$ and $0 \le \delta < \left(\frac{n}{n+1}\right)^n$. For $u \in \mathbb{R}^n$ and $a \in \mathbb{R}$, $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = a\}$ is the hyperplane orthogonal to u and $H^+ = \{x \in \mathbb{R}^n : \langle x, u\rangle \ge a\}$ and $H^- = \{x \in \mathbb{R}^n : \langle x, u\rangle \le a\}$ are the two half spaces determined by H. Then the (convex) floating body K_δ [34] of K is the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume at most $\delta|K|$ from K,

$$K_\delta = \bigcap_{\{H : |H^- \cap K| \le \delta|K|\}} H^+. \tag{11}$$

Clearly, $K_0 = K$ and $K_\delta \subseteq K$ for all $\delta \ge 0$. The condition $\delta < \left(\frac{n}{n+1}\right)^n$ insures that $g(K) \in \mathrm{int}(K_\delta) \ne \emptyset$ (see [34]). Moreover, for all invertible affine maps T, one has

$$(T(K))_\delta = T\left(K_\delta\right). \tag{12}$$

To prove that $K \to K_\delta$ is continuous from $\mathcal{K}_n$ to $\mathcal{K}_n$, we need some notation. For $u \in S^{n-1}$, we define $a_{\delta,K}(u)$ to be unique real number such that

$$\mathrm{vol}_n\left(\{x \in K : \langle x, u\rangle \ge a_{\delta,K}(u)\}\right) = \delta\ \mathrm{vol}_n(K). \tag{13}$$

Then

$$K_\delta = \bigcap_{u \in S^{n-1}} \{x \in K : \langle x, u\rangle \le a_{\delta,K}(u)\}.$$

Lemma 4. *Let $K \in \mathcal{K}_n$, $u \in S^{n-1}$, $0 < \delta < \left(\frac{n}{n+1}\right)^n$ and $f(t) = |\{x \in K : \langle x, u\rangle = t\}|$. Let $a \in \mathbb{R}$ satisfy $\int_a^{+\infty} f(t)dt = \delta \int_{-\infty}^{+\infty} f(t)dt$. Then one has*

$$f(a) \ge \delta^{\frac{n-1}{n}} \max_{t \in \mathbb{R}} f(t).$$

Proof. By the Brunn-Minkowski theorem (see [4, 31]), $f^{\frac{1}{n-1}}$ is concave on $\{f > 0\}$. Put $M = f(m) = \max_{t\in\mathbb{R}} f(t)$.

We suppose first that $m < a$. Let g be the affine function on $\mathbb{R}$ such that $g^{n-1}(m) = f(m)$ and $g^{n-1}(a) = f(a)$. As $f^{\frac{1}{n-1}}$ is concave on $\{f \ne 0\}$, one has $g^{n-1} \le f$ on $[m, a]$ and $g^{n-1} \ge f$ on $\{f \ne 0\} \setminus [m, a]$. Thus there exists $c \le m$ and $d \ge a$, such that $g(c) > 0$, $g(d) > 0$,

$$\int_c^d g^{n-1}(t)dt = \int_{-\infty}^{+\infty} f(t)dt \quad \text{and} \quad \int_a^d g^{n-1}(t)dt = \int_a^{+\infty} f(t)dt.$$

Let $g_1 = g\mathbf{1}_{[c,d]}$. Since g is non increasing on $[c,d]$, $g_1^{n-1}(c) \geq M = f(m)$. Moreover, by construction, $g_1^{n-1}(a) = f(a)$. We replace now g_1 with a new function g_2 that is affine on its support $[c', d']$, $c' \leq a \leq d'$, and satisfies $g_2^{n-1}(a) = f(a)$, $g_2^{n-1}(d') = 0$,

$$\int_{c'}^{d'} g_2^{n-1}(t)dt = \int_{-\infty}^{+\infty} f(t)dt \quad \text{and} \quad \int_a^{d'} g_2^{n-1}(t)dt = \int_a^{+\infty} f(t)dt.$$

One still has $g_2^{n-1}(a) = f(a)$ and clearly $g_2^{n-1}(c') \geq g_1^{n-1}(c) \geq M$. Now, an easy computation gives

$$f(a) = g_2^{n-1}(a) = \delta^{\frac{n-1}{n}}, \quad g_2^{n-1}(c') \geq \delta^{\frac{n-1}{n}} M.$$

We suppose next that $m \geq a$. The same reasoning, with $1-\delta$ instead of δ, gives $f(a) \geq (1-\delta)^{\frac{n-1}{n}} M$. Since $0 < \delta < \frac{1}{2}$, the statement follows. $\square$

Proposition 4. *Let $0 < r \leq R < \infty$ and let $K \in \mathcal{K}_n$ satisfy, $rB_2^n \subseteq K \subseteq RB_2^n$. Let $0 < \delta < \frac{1}{2}$ and $\eta > 0$. There there exists $\varepsilon > 0$ (depending only on r, R, n, δ) such that, whenever a convex body L satisfies $d_H(K, L) \leq \varepsilon$, one has for every $u \in S^{n-1}$*

$$a_{\delta,K}(u) - \eta \leq a_{\delta,L}(u) \leq a_{\delta,K}(u) + \eta.$$

Proof. Let $\rho > 0$. With the hypothesis on K, we may choose $\varepsilon > 0$ small enough such that whenever $d_H(K, L) \leq \varepsilon$, then $(1-\rho)K \subseteq L \subseteq (1+\rho)K$. Fix $u \in S^{n-1}$ and define

$$f_K(t) = \mathrm{vol}_{n-1}\left(\{x \in K : \langle x, u\rangle = t\}\right) \text{ and } f_L(t) = \mathrm{vol}_{n-1}\left(\{x \in L : \langle x, u\rangle = t\}\right).$$

Then $a_{\delta,K} := a_{\delta,K}(u)$ and $a_{\delta,L} := a_{\delta,L}(u)$ satisfy

$$\int_{a_{\delta,K}}^{+\infty} f_K(t)dt = \delta|K| \quad \text{and} \quad \int_{a_{\delta,L}}^{+\infty} f_L(t)dt = \delta|L|.$$

Let $\theta > 0$. For $\rho > 0$ small enough one has,

$$|K\Delta L| \leq \left((1+\rho)^n - (1-\rho)^n\right)|K| \leq \theta.$$

For such a ρ one has also

$$\int_{\mathbb{R}} |f_K(t) - f_L(t)|dt \quad \leq \quad \int_{\mathbb{R}} \mathrm{vol}_{n-1}\left(\{x \in K\Delta L : \langle x, u\rangle = t\}\right)dt = |K\Delta L| \leq \theta,$$

so that

$$\left|\int_{[a_{\delta,K}, a_{\delta,L}]} f_K(t)dt\right| \leq \left|\int_{a_{\delta,K}}^{+\infty} f_K(t)dt - \int_{a_{\delta,L}}^{+\infty} f_L(t)dt\right| + \int_{a_{\delta,L}}^{+\infty} |f_K(t) - f_L(t)|dt \leq 2\theta.$$

For $\alpha > 0$ given, let $\theta = \frac{\alpha|K|}{2}$. Then

$$\left|\int_{a_{\delta,L}}^{+\infty} f_K(t)dt - \delta|K|\right| \leq \alpha\ \delta|K|.$$

For some $\beta \in [-\alpha, \alpha]$, one has hence that $a_{\delta,L} = a_{(1+\beta)\delta,K}$. Concavity of $f_K^{\frac{1}{n-1}}$ on $\{f_K \neq 0\}$ implies that

$$\left| \int_{[a_{\delta,K}, a_{\delta,L}]} f_K(t) dt \right| \geq |a_{\delta,K} - a_{\delta(1+\beta),K}| \min \left(f_K(a_{\delta,K}), f_K(a_{(1+\beta)\delta,K}) \right).$$

If $M = \max_{t \in \mathbb{R}} f_K(t)$, we get by Lemma 4,

$$\min \left(f_K(a_{\delta,K}), f_K(a_{(1+\alpha)\delta} \right) \geq \left(\min(1+\beta, 1)\delta \right)^{\frac{n-1}{n}} \geq \left((1-\alpha)\delta \right)^{\frac{n-1}{n}} M.$$

Since $K \subseteq RB_2^n$, we estimate M from above by

$$M \leq \gamma_n = R^{n-1} |B_2^{n-1}|,$$

which is an upper bound independent of u. It follows that if $\alpha > 0$ is small enough, then

$$|a_{\delta,K} - a_{\delta,L}| \leq 2 \; \theta \; \gamma_n \; \left((1-\alpha)\delta \right)^{-\frac{n-1}{n}} \leq \eta.$$

□

The next proposition shows that the map $K \mapsto K_\delta$ as defined in (11), is an affine invariant set mapping.

Proposition 5. *For $0 < \delta < \left(\frac{n}{n+1}\right)^n$, the mapping $K \mapsto K_\delta$ is is an affine invariant set mapping from $\mathcal{K}_n$ to $\mathcal{K}_n$.*

Proof. We take $0 < \delta < \left(\frac{n}{n+1}\right)^n$ so that $K_\delta \neq \emptyset$. It is clear that $K \to K_\delta$ is an affine invariant mapping and it is clear that $g(K) \in K_\delta$. We now fix a body $K \in \mathcal{K}_n$ and we verify the continuity of the mapping $K \to K_\delta$ at K. We may suppose that 0 is the center of mass of K. For some $0 < r \leq R < \infty$, one has

$$rB_2^n \subseteq K_\delta \subseteq K \subseteq RB_2^n.$$

By the choice of δ, $a_{\delta,K}(u) > 0$ for every $u \in S^{n-1}$, where $a_{\delta,K}(u)$ is as in (13). Let $\eta, \eta' > 0$ satisfy $\eta' \leq \eta r \leq \eta \; \min_{u \in S^{n-1}} a_{\delta,K}(u)$. We use the notation of the preceding proposition to find $\varepsilon > 0$ such that for any L with $d_H(K, L) \leq \varepsilon$, one has

$$a_{\delta,K}(u) - \eta' \leq a_{\delta,L}(u) \leq a_{\delta,K}(u) + \eta',$$

or

$$(1-\eta) a_{\delta,K}(u) \leq a_{\delta,L}(u) \leq (1+\eta) a_{\delta,K}(u),$$

whence

$$(1-\eta) K_\delta \subseteq L_\delta \subseteq (1+\eta) K_\delta.$$

Since $rB_2^n \subseteq K_\delta \subseteq RB_2^n$, it then follows that, given $\rho > 0$, for $\eta > 0$ small enough, one has $d_H(L_\delta, K_\delta) \leq \rho$. □

As a corollary, we obtain new affine invariant points.

Corollary 1. *Let $0 < \delta < \left(\frac{n}{n+1}\right)^n$ and let $p : \mathcal{K}_n \to \mathbb{R}^n$ be an affine invariant point. Then $K \to p(K_\delta)$ is also an affine invariant point. In particular, for the centroid g, $K \mapsto g(K \setminus K_\delta)$ is an affine invariant point.*

Proof. Affine invariance follows from Remark (i) after Definition 2 and continuity from Proposition 5. The second statement follows now from the trivial identity

$$g(K) = \frac{|K_\delta|}{|K|} g(K_\delta) + \frac{|K \setminus K_\delta|}{|K|} g(K \setminus K_\delta),$$

which gives

$$g(K \setminus K_\delta) = \frac{|K|}{|K \setminus K_\delta|}\ g(K) - \frac{|K_\delta|)}{|K \setminus K_\delta|}\ g(K_\delta),$$

as an affine combination of continuous affine invariant points. □

The next lemma is key for many of the proofs that will follow.

Lemma 5. *Let $m \geq n+1$ and for $1 \leq i \leq m$, let $v_i \in \mathbb{R}^n$ be the vertices of a polytope P in $\mathcal{K}_n$. Then for all $\varepsilon > 0$ there is $z \in P$ with $\|v_1 - z\| \leq \varepsilon$ and $0 < r \leq \varepsilon$ such that $B_2^n(z, r) \subset P$ and if $K = \mathrm{conv}\,(B_2^n(z, r), v_2, \dots, v_m)$, then K satisfies*

(i) $K \subseteq P$, $v_2, \dots, v_m$ are extreme points of K and $d_H(K, P) \leq \varepsilon$.

(ii) For sufficiently small δ, $\|v_1 - g\,(K \setminus K_\delta)\,\| \leq 2\varepsilon$

Proof. There exists a hyperplane H that strictly separates v_1 and $\{v_2, \dots, v_m\}$, such that for all $x \in H^- \cap P$ we have that $\|x - v_1\| < \varepsilon$. Let $z \in \mathrm{int}(H^-) \cap \mathrm{int}(P)$. Then there exists $0 < r \leq \varepsilon$ such that $B_2^n(z, r) \subseteq H^- \cap P$. Let $K = \mathrm{conv}\,(B_2^n(z, r), v_2, \dots, v_m)$.

(i) By construction of K, $v_2 \dots, v_m$ are extreme points of K. Also, for all $x \in K \cap H^-$, one has $\|x - v_1\| < \varepsilon$. Therefore $K \subseteq P \subseteq K + \varepsilon B_2^n$ and thus $d_H(K, P) \leq \varepsilon$.

(ii) We have for $\delta > 0$,

$$\begin{aligned} g\,(K \setminus K_\delta) &= \frac{|(K \cap H^+) \setminus K_\delta|}{|K \setminus K_\delta|} g\left(\left(K \cap H^+\right) \setminus K_\delta\right) \\ &+ \frac{|(K \cap H^-) \setminus K_\delta|}{|K \setminus K_\delta|} g\left(\left(K \cap H^-\right) \setminus K_\delta\right). \end{aligned} \tag{14}$$

Since $g\,((K \cap H^-) \setminus K_\delta) \in \mathrm{int}\,(K) \cap \mathrm{int}\,(H^-)$, one has

$$\|v_1 - g\left(\left(K \cap H^-\right) \setminus K_\delta\right)\| \leq \varepsilon.$$

Observe that ∂K contains a cap of $\partial B(z, r)$, so that

$$C = \int_{\partial K} \kappa_K^{\frac{1}{n+1}}\, d\mu_K > 0.$$

By Theorem 4, one has for δ sufficiently small,

$$|K \setminus K_\delta| = |K| - |K_\delta| \geq \frac{C}{2c_n}\,(\delta |K|)^{\frac{2}{n+1}}. \tag{15}$$

Let $R = \max\{\|x\| : x \in P\}$. As the Gauss curvature is equal to 0 everywhere on the boundary $\partial (K \cap H^+)$, again by Theorem 4, one has for sufficiently small δ,

$$c_n \frac{|K \cap H^+| - \left|(K \cap H^+)_{\frac{\delta|K|}{|K \cap H^+|}}\right|}{(\delta|K|)^{\frac{2}{n+1}}} \le \frac{C\varepsilon}{4R}.$$

As

$$\left(K \cap H^+\right) \setminus K_\delta \subseteq \left(K \cap H^+\right) \setminus \left(K \cap H^+\right)_{\frac{\delta|K|}{|K \cap H_2^+|}},$$

we get

$$\left|\left(K \cap H^+\right) \setminus K_\delta\right| \le \frac{C\varepsilon}{4Rc_n} \ (\delta|K|)^{\frac{2}{n+1}}. \tag{16}$$

It folllows from (15) and (16) that for δ small enough one has

$$\frac{|(K \cap H^+) \setminus K_\delta|}{|K \setminus K_\delta|} \le \frac{\varepsilon}{R}. \tag{17}$$

We get thus from (14) and (17)

$$\begin{aligned}
&\|g\left(K \setminus K_\delta\right) - g\left(\left(K \cap H^-\right) \setminus K_\delta\right)\| \\
&= \frac{|(K \cap H^+) \setminus K_\delta|}{|K \setminus K_\delta|} \|g\left(\left(K \cap H^+\right) \setminus K_\delta\right) - g\left(\left(K \cap H^-\right) \setminus K_\delta\right)\| \\
&\le \frac{|(K \cap H^+) \setminus K_\delta|}{|K \setminus K_\delta|} \left(\|g\left(\left(K \cap H^-\right) \setminus K_\delta\right)\| + \|g\left(\left(K \cap H^+\right) \setminus K_\delta\right)\|\right) \\
&\le \frac{\varepsilon}{2R}(R + R) \le \varepsilon
\end{aligned}$$

Altogether,

$$\|v_1 - g\left(K \setminus K_\delta\right)\| \le \|v_1 - g\left(\left(K \cap H^-\right) \setminus K_\delta\right)\| + \|g\left(K \setminus K_\delta\right) - g\left(\left(K \cap H^-\right) \setminus K_\delta\right)\| \le 2\varepsilon.$$

□

3.2 Proof of Theorem 1: $\mathfrak{P}_n$ is infinite dimensional.

Here, we answer in the negative Grünbaum's question whether there exists a finite basis for $\mathfrak{P}_n$, i.e. affine invariant points $p_i \in \mathfrak{P}_n$, $1 \le i \le l$, such that every $p \in \mathfrak{P}_n$ can be written as

$$p = \sum_{i=1}^{l} \alpha_i p_i, \quad \alpha_i \in \mathbb{R} \text{ for all } i \quad \text{and} \quad \sum_{i=1}^{l} \alpha_i = 1.$$

Recall that $\mathfrak{P}_n$ is an affine subspace of $C(\mathcal{K}_n, \mathbb{R}^n)$, the continuous functions on $\mathcal{K}_n$ with values in $\mathbb{R}^n$ and that we denote by $V\mathfrak{P}_n$ the subspace parallel to $\mathfrak{P}_n$. Thus, with the centroid g,

$$V\mathfrak{P}_n = \mathfrak{P}_n - g. \tag{18}$$

The dimension of $\mathfrak{P}_n$ is the dimension of $V\mathfrak{P}_n$. We introduce a norm on $V\mathfrak{P}_n$,

$$\|v\|_{\mathfrak{P}} = \sup_{\substack{K \in \mathcal{K}_n \\ B_2^n \subseteq K \subseteq nB_2^n}} \|v(K)\|, \qquad \text{for } v \in V\mathfrak{P}_n. \tag{19}$$

Please note that the set $\{K \in \mathcal{K}_n : B_2^n \subseteq K \subseteq nB_2^n\}$ is a compact subset of $(\mathcal{K}_n, d_H)$. Therefore (19) is well defined and it is a norm: $v = p - g \neq 0$ implies that there is C with $v(C) \neq 0$. By John's theorem (e.g., [40]), there is an affine, invertible map T with $B_2^n \subseteq T(C) \subseteq nB_2^n$. Thus,

$$v(T(C)) = (p - g)(T(C)) = p(T(C)) - g(T(C)) = T(p(C)) - T(g(C)).$$

Since $T = S + x_0$, where S is a linear map,

$$v(T(C)) = S(p(C) - g(C)) = S((p - g)(C)) \neq 0.$$

Hence

$$\|v\|_{\mathfrak{P}} \geq \|v(T(C))\| > 0.$$

For the proof of Theorem 1 and Theorem 3, we will make use of the following theorem by Schütt and Werner [34]. There, μ_K is the usual surface measure on ∂K and for $x \in \partial K$, $\kappa(x)$ is the generalized Gauss curvature at x, which is defined μ_K almost everywhere.

Theorem 4. *[34] Let K be a convex body in $\mathbb{R}^n$. Then, if $c_n = 2\left(\frac{|B^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}$, one has*

$$c_n \lim_{\delta \to 0} \frac{|K| - |K_\delta|}{(\delta |K|)^{\frac{2}{n+1}}} = \int_{\partial K} \kappa^{\frac{1}{n+1}}(x) \, d\mu_K(x).$$

Proof of Theorem 1. We show that the closed unit ball of $V\mathfrak{P}_n$ is not compact. For $K \in \mathcal{K}_n$ and $\delta > 0$, let K_δ be the convex floating body of K. Let g be the centroid and let $g_\delta : \mathcal{K}_n \to \mathbb{R}^n$ be the affine invariant point given by

$$g_\delta(K) = g(K \setminus K_\delta).$$

The set of vectors $\{v_\delta = g_\delta - g \colon \ \delta > 0\}$ is bounded. Indeed, since $g(K) \in K$ and $g_\delta(K) \in K$

$$\|v_\delta\|_{\mathfrak{P}} \leq \sup_{\substack{K \in \mathcal{K}_n \\ B_2^n \subseteq K \subseteq nB_2^n}} (\|g(K)\| + \|g_\delta(K)\|) \leq 2n.$$

Now we show that there is a sequence δ_j, $j \in \mathbb{N}$, of strictly positive numbers such that for all j, k with $j \neq k$

$$\frac{1}{10} \leq \left\|v_{\delta_j} - v_{\delta_k}\right\|_{\mathfrak{P}},$$

i.e.

$$\sup_{\substack{K \in \mathcal{K}_n \\ B_2^n \subseteq K \subseteq nB_2^n}} \left\|v_{\delta_j}(K) - v_{\delta_k}(K)\right\| \geq \frac{1}{10}.$$

As K we consider the union of the cylinder $D = [-1, 1] \times B_2^{n-1}$ and a cap of height h of a Euclidean ball,

$$K(h) = D \cup \ C(h), \tag{20}$$

where, with $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$,

$$C(h) = \left(\frac{h^2 + 2h - 1}{2h} e_1 + \frac{1 + h^2}{2h} B_2^n\right) \cap \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 1\}.$$

We show that there are two sequences δ_j, $j \in \mathbb{N}$, and h_j, $j \in \mathbb{N}$, such that for all $j = 2, 3, \dots$ and all i with $1 \leq i < j$,

$$\|v_{\delta_i}(K(h_j)) - v_{\delta_j}(K(h_j))\| \geq \frac{1}{4}.$$

We show this by induction. We choose h_1 so small that for all h with $0 < h \leq h_1$

$$\|g(K(h))\| \leq \frac{1}{8}.$$

This is possible because $K(h) \to D$ for $h \to 0$ and because, by Corollary 1, $g(K(h)) \to g(D) = 0$.

Then we choose δ_1 so small that

$$\|g_{\delta_1}(K_{h_1})\| \geq \frac{3}{4}. \tag{21}$$

This is possible by the same argument as in the proof of Lemma 5: Let H be the hyperplane such that

$$K(h) \cap H^- = C(h) \qquad \text{and} \qquad K(h) \cap H^+ = D.$$

Then as in (14),

$$g_\delta(K(h)) = \frac{|D \setminus K(h)_\delta|}{|K(h) \setminus K(h)_\delta|} g\left(D \setminus K(h)_\delta\right) + \frac{|C(h) \setminus K(h)_\delta|}{|K(h) \setminus K(h)_\delta|} g\left(C(h) \setminus K(h)_\delta\right).$$

Since

$$g(D \setminus K(h)_\delta) \in D \qquad \text{and} \qquad g(C(h) \setminus K(h)_\delta) \in C(h)$$

we get

$$\|g(D \setminus K(h)_\delta)\| \leq \sqrt{2} \qquad \text{and} \qquad \|g(C(h) \setminus K(h)_\delta\| \geq 1.$$

Therefore, by triangle inequality,

$$\|g_\delta(K(h))\| \geq \frac{|C(h) \setminus K(h)_\delta|}{|K(h) \setminus K(h)_\delta|} - \sqrt{2}\frac{|D \setminus K(h)_\delta|}{|K(h) \setminus K(h)_\delta|}. \tag{22}$$

By Theorem 4, for δ small enough, with

$$\alpha(h) = \int_{\partial K(h)} \kappa_{K(h)}^{\frac{1}{n+1}} d\mu_{K(h)},$$

we get as in (15),

$$|K(h) \setminus K(h)_\delta| \geq \frac{(\delta |K(h)|)^{\frac{2}{n+1}}}{2c_n} \int_{\partial K(h)} \kappa_{K(h)}^{\frac{1}{n+1}} d\mu_{K(h)} = \frac{(\delta |K(h)|)^{\frac{2}{n+1}}}{2c_n} \alpha(h).$$

Moreover,

$$|D \setminus K(h)_\delta| \leq |D \setminus D_\delta|.$$

We apply Theorem 4 to D. The curvature of the boundary of D is almost everywhere 0. Therefore, for δ sufficiently small

$$\frac{|D \setminus K(h)_\delta|}{(\delta|D|)^{\frac{2}{n+1}}}$$

is as small as we want. Altogether, there is δ_1 such that (21) holds.

Suppose that we have already chosen $h_1, \ldots, h_j$ and $\delta_1, \ldots, \delta_j$. We choose h_{j+1} so small that for all i with $1 \leq i \leq j$

$$\|g_{\delta_i}(K_{h_{j+1}})\| \leq \frac{1}{8}.$$

Again, this is possible as $K(h) \to D$ for $h \to 0$ and as $g_\delta(K(h)) \to g_\delta(D) = 0$ by Corollary 1. Now we choose δ_{j+1} so small that

$$\|g_{\delta_{j+1}}(K_{h_{j+1}})\| \geq \frac{3}{4}.$$

The argument here is the same as for choosing δ_1. $\square$

3.3 Proof of Theorem 2.

It was also asked by Grünbaum [9] if for every $K \in \mathcal{K}_n$,

$$\mathfrak{P}_n(K) = \mathfrak{F}_n(K),$$

where $\mathfrak{F}_n(K) = \{x \in \mathbb{R}^n : Tx = x, \text{ for all affine } T \text{ with } TK = K\}$. Observe that it is clear that $\mathfrak{P}_n(K) \subseteq \mathfrak{F}_n(K)$. We will prove that $\mathfrak{P}_n(K) = \mathfrak{F}_n(K)$, if $\mathfrak{P}_n(K)$ is $(n-1)$-dimensional. To do so, we, again, first need to define new affine invariant set mappings.

Actually, in the proof of Theorem 2 we show that the group of isometries of K equals

$$\{I_n, S\} = \{T : \mathbb{R}^n \to \mathbb{R}^n \text{ affine one to one, } \ TK = K\},$$

where S is reflection about a hyperplane, i.e. $S : \mathbb{R}^n \to \mathbb{R}^n$ is bijective and there is a hyperplane H and a direction $\xi \notin H$ such that $S(h + t\xi) = h - t\xi$ for all $h \in H$.

Lemma 6. *Let $p \in \mathfrak{P}_n$ and let g be the centroid. For $0 < \varepsilon < 1$, define $A_{p,\epsilon}, B_{p,\epsilon} : \mathcal{K}_n \to \mathcal{K}_n$ by*

$$A_{p,\epsilon}(K) = \left\{ x \in K \,\middle|\, \langle x, p((K - g(K))^\circ)\rangle \geq \sup_{y \in K} \langle y, p((K - g(K))^\circ)\rangle - \varepsilon \right\}.$$

and

$$B_{p,\epsilon}(K) = \left\{ x \in K \,\middle|\, \langle x, p((K - g(K))^\circ)\rangle \leq \inf_{y \in K} \langle y, p((K - g(K))^\circ)\rangle + \varepsilon \right\}.$$

Then $A_{p,\epsilon}$ and $B_{p,\epsilon}$ are affine invariant set maps.

Please note that $0 \in \mathfrak{P}_n((K - g(K))^\circ)$ since 0 is the Santaló point of $(K - g(K))^\circ$. Therefore, $\mathfrak{P}_n((K - g(K))^\circ)$ is a subspace of $\mathbb{R}^n$.

Proof. Let T be an invertible, affine map and $T = S + a$ its decomposition in a linear map S and a translation a. Then for any convex body C that contains 0 in its interior,

$$(S(C))^\circ = S^{*-1}(C^\circ).$$

Moreover,

$$\begin{aligned}p((T(K)-g(T(K)))^\circ) &= p((S(K-g(K)))^\circ)\\ &= p(S^{*-1}((K-g(K))^\circ)) = S^{*-1}(p((K-g(K))^\circ)).\end{aligned}$$

Since $S^{*-1*}=S^{-1}$,

$$\begin{aligned}&A_{p,\epsilon}(T(K))\\ &=\left\{x\in T(K)\ \middle|\ \langle x,p((T(K)-g(T(K)))^\circ)\rangle \geq \sup_{y\in T(K)}\langle y,p((T(K)-g(T(K)))^\circ)\rangle-\varepsilon\right\}\\ &=\left\{x\in T(K)\ \middle|\ \langle S^{-1}x,p((K-g(K))^\circ)\rangle \geq \sup_{y\in T(K)}\langle S^{-1}y,p((K-g(K))^\circ)\rangle-\varepsilon\right\}\end{aligned}$$

and one verifies easily that $A_{p,\epsilon}(T(K))=T(A_{p,\epsilon}(K))$. Please note that $A_{p,\epsilon}(K)$ is convex, compact and nonempty.

We have to show that $A_{p,\epsilon}$ and $B_{p,\epsilon}$ are continuous. We consider $A_{p,\epsilon}$ first.

We shall show that for every K and $\alpha>0$ there is $\beta>0$ such that for all C with $d_H(K,C)<\beta$ we have

$$d_H(A_{p,\epsilon}(K),A_{p,\epsilon}(C))<\alpha.$$

We consider two cases. First,

$$p((K-g(K))^\circ)=0.$$

We claim that there is β such that for all C with $d_H(K,C)<\beta$

$$A_{p,\epsilon}(C)=C.$$

We verify this.

$$\begin{aligned}&\langle x,p((C-g(C))^\circ)\rangle\\ &\geq \inf_{z\in K_j}\langle z,p((C-g(C))^\circ)\rangle \geq -\sup_{z\in C}\|z\|_2\|p((C-g(C))^\circ)\|_2\\ &\geq \sup_{z\in C}\langle z,p((C-g(C))^\circ)\rangle - 2\sup_{z\in K_j}\|z\|_2\|p((C-g(C))^\circ)\|_2\end{aligned}$$

Since $p((K-g(K))^\circ)=0$ and since p and g are continuous, we may choose β so small that

$$2\sup_{z\in K_j}\|z\|_2\|p((C-g(C))^\circ)\|_2<\epsilon.$$

Now we consider the case

$$p((K-g(K))^\circ)\neq 0.$$

Let K and C be convex bodies with $d_H(K,C)=\rho$. Moreover, we denote

$$p_K=p(K-g(K))^\circ) \qquad \text{and} \qquad p_C=p(C-g(C))^\circ).$$

We have

$$\langle x,p_K\rangle=\langle x,p_C\rangle+\langle x,p_K-p_C\rangle\leq\langle x,p_C\rangle+\|x\|_2\|p_K-p_C\|_2$$

and

$$\sup_{y\in K}\langle y, p_K\rangle = \sup_{y\in K}(\langle y, p_C\rangle + \langle y, p_K - p_C\rangle) \geq \sup_{y\in K}\langle y, p_C\rangle - \sup \|y\|_2\|p_K - p_C\|_2.$$

Therefore, for all x with

$$\langle x, p_K\rangle \geq \sup_{y\in K}\langle y, p_K\rangle - \epsilon$$

we have

$$\langle x, p_C\rangle \geq \sup_{y\in K}\langle y, p_C\rangle - \sup_{y\in K}\|y\|_2\|p_K - p_C\|_2 - \|x\|_2\|p_K - p_C\|_2 - \epsilon.$$

Since $d_H(K, C) = \rho$ we have $C \subseteq K + \rho B_2^n$ and

$$\sup_{y\in C}\langle y, p_C\rangle \leq \sup_{z\in K+\rho B_2^n}\langle z, p_C\rangle \leq \sup_{y\in K}\langle y, p_C\rangle + \rho\|p_C\|_2.$$

Therefore

$$\sup_{y\in C}\langle y, p_C\rangle - \rho\|p_C\|_2 \leq \sup_{y\in K}\langle y, p_C\rangle.$$

Altogether, we have for all x with $\langle x, p_K\rangle \geq \sup_{y\in K}\langle y, p_K\rangle - \epsilon$

$$\langle x, p_C\rangle \geq \sup_{y\in C}\langle y, p_C\rangle - \rho\|p_C\|_2 - \sup_{y\in K}\|y\|_2\|p_K - p_C\|_2 - \|x\|_2\|p_K - p_C\|_2 - \epsilon.$$

Therefore,

$$A_{p,\epsilon}(K) \subseteq A_{p,\delta}(C)$$

where

$$\delta \leq \epsilon + \rho\|p_C\|_2 + \sup_{y\in K}\|y\|_2\|p_K - p_C\|_2 + \|x\|_2\|p_K - p_C\|_2.$$

This means that for all $\alpha > 0$ there is $\beta > 0$ such that for all C with $d_H(K, C) \leq \beta$

$$A_{p,\epsilon}(K) \subseteq A_{p,\epsilon+\alpha}(C).$$

Now it is left to observe that for every $\eta > 0$ and every $\delta > 0$ there is $\epsilon > 0$ with $\epsilon < \delta$ and

$$A_{p,\delta}(C) \subseteq A_{p,\epsilon}(C) + \eta B_2^n.$$

□

Lemma 7. *Let $K \in \mathcal{K}_n$ and let $P : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $\mathfrak{P}_n((K - g(K))^\circ)$. Then the restriction of P to the subspace $\mathfrak{P}_n(K - g(K))$ is an isomorphismn between $\mathfrak{P}_n(K - g(K))$ and $\mathfrak{P}_n((K - g(K))^\circ)$.*

In particular,

$$\dim(\mathfrak{P}_n(K - g(K))) = \dim(\mathfrak{P}_n((K - g(K)))^\circ).$$

Proof. On the hyperplane $\mathfrak{P}_n((K - g(K))^\circ)$, $P(K - g(K))$ has an interior point. This holds because otherwise, by Fubini, $\mathrm{vol}_n(K) = 0$.

Let $k = \dim(\mathfrak{P}_n((K - g(K)))^\circ)$. We choose $u_1 \in \mathfrak{P}_n((K - g(K))^\circ)$. Then $g(A_{u_1,\epsilon_1})$ is a proper affine invariant point. Now we choose $u_2 \in \mathfrak{P}_n((K - g(K))^\circ)$ that is orthogonal to $P(g(A_{u_1,\epsilon_1}))$. Then $P(g(A_{u_1,\epsilon_1}))$ and $P(g(A_{u_2,\epsilon_2}))$ are linearly independent.

Eventually,

$$P(g(A_{u_1,\epsilon_1})), \dots, P(g(A_{u_k,\epsilon_k}))$$

are linearly independent, and therefore

$$g(A_{u_1,\epsilon_1}), \dots, g(A_{u_k,\epsilon_k})$$

are linearly independent. Therefore,

$$\dim(\mathfrak{P}_n((K-g(K)))^\circ) \leq \dim(\mathfrak{P}_n(K-g(K))).$$

Now we interchange the roles of $\mathfrak{P}_n(K-g(K))$ and $\mathfrak{P}_n((K-g(K))^\circ)$ and get the inverse inequality.

Let Q denote the restriction of P to the subspace $\mathfrak{P}_n(K-g(K))$. $g(A_{u_1,\epsilon_1}), \dots, g(A_{u_k,\epsilon_k})$ is a basis of $\mathfrak{P}_n(K-g(K))$ and $P(g(A_{u_1,\epsilon_1})), \dots, P(g(A_{u_k,\epsilon_k}))$ is a basis of $\mathfrak{P}_n((K-g(K)))^\circ$. Q is a bijection between the two bases, thus Q is an isomorphism. □

Lemma 8. *Let $K \in \mathcal{K}_n$. Then for every point x from the relative interior of $K \cap \mathfrak{P}_n(K)$ there is a proper affine invariant point q with $q(K) = x$.*

Proof. We use the same notation as in Lemma 7 and its proof. We may assume that $g(K) = 0$. Suppose that there is an interior point x of $\mathfrak{P}_n(K) \cap K$ in the hyperplane $\mathfrak{P}_n(K)$ for which there is no proper affine invariant point q with $q(K) = x$. The set

$$\{p(K) | p \text{ is a proper affine invariant point}\}$$

is convex. $P : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto $\mathfrak{P}_n(K^\circ)$. Then $P(\mathfrak{P}_n(K) \cap K)$ is a convex set in the hyperplane $\mathfrak{P}_n(K^\circ)$. Since P is an isomorphism between the hyperplanes $\mathfrak{P}_n(K)$ and $\mathfrak{P}_n(K^\circ)$ we have

$$P(x) \notin P(\{p(K) | p \text{ is a proper affine invariant point}\}).$$

Moreover, $P(x)$ is an interior point of $P(\mathfrak{P}_n(K) \cap K)$. By the Hahn-Banach theorem there is $u \in \mathfrak{P}_n(K^\circ)$ such that for all proper affine invariant points p we have

$$\langle u, x \rangle \geq \langle u, P(p(K)) \rangle.$$

On the other hand, there is an affine invariant point q with $q(K^\circ) = u$. Then $g \circ A_{u,\langle u,x\rangle}$ is a proper affine invariant point with

$$\langle u, x \rangle < \langle u, g \circ A_{q,\langle u,x\rangle} \rangle,$$

which is a contradiction. □

Lemma 9. *Let $K \in \mathcal{K}_n$ and suppose that $\dim(\mathfrak{P}_n(K)) = n-1$. Then $S : \mathbb{R}^n \to \mathbb{R}^n$ with*

$$S(y + x) = y - x$$

for all $y \in \mathfrak{P}_n(K-g(K))$ and $x \in \mathfrak{P}_n((K-g(K))^\circ)^\perp$ is a linear map such that

$$S(K - g(K)) = K - g(K).$$

Proof. By Lemma 7, the orthogonal projection onto $\mathfrak{P}_n((K-g(K))^\circ)$ restricted to $\mathfrak{P}_n(K-g(K))$ is an isomorphism. Therefore,

$$\mathbb{R}^n = \mathfrak{P}_n(K-g(K)) \oplus \mathfrak{P}_n((K-g(K))^\circ)^\perp.$$

By Lemma 8 for every $y \in \mathfrak{P}_n(K-g(K)) \cap \operatorname{int}(K)$ there is a proper affine invariant point q with $y = q(K)$. Let $u_1, \dots, u_{n-1}$ be an orthonormal basis in $\mathfrak{P}_n((K-g(K))^\circ)$. The map $A_\epsilon : \mathcal{K}_n \to \mathcal{K}_n$ with

$$A_\epsilon(K) = \bigcap_{i=1}^{n-1} \{x \in K | \langle q(K), u_i \rangle - \epsilon \le \langle x, u_i \rangle \le \langle q(K), u_i \rangle + \epsilon\}$$

is an affine invariant set map. Moreover,

$$\lim_{\epsilon \to 0} A_\epsilon(K) = K \cap (q(K) + \mathfrak{P}_n((K-g(K))^\circ)^\perp)$$

in the Hausdorff metric. $g \circ A_\epsilon$ is a proper affine invariant point. Since all affine invariant points are elements of $\mathfrak{P}_n(K)$

$$\lim_{\epsilon \to 0} (g \circ A_\epsilon)(K) = q(K).$$

On the other hand, $q(K)$ is the midpoint of $K \cap (q(K) + \mathfrak{P}_n((K-g(K))^\circ)^\perp)$. $\square$

Proof of Theorem 2. Theorem 2 now follows immediately from Lemma 9. Indeed, Lemma 9 provides a map $T = S - S(g(K)) + g(K)$ with $T(K) = K$ and such that for all $z \in \mathfrak{P}_n(K)$ and for all $x \in \mathfrak{P}_n((K-g(K))^\circ)^\perp$,

$$T(z+x) = z - x.$$

Consequently, if $w \notin \mathfrak{P}_n(K)$, then $T(w) \neq w$, which means that the complement of $\mathfrak{P}_n(K)$ is contained in the complement of $\mathfrak{F}_n(K)$.

Remark. As a byproduct of the preceding results, it can be proved that if $K \in \mathcal{K}_n$ satisfies $\mathfrak{P}_n(K) = \mathbb{R}^n$ and if $\mathfrak{S}_n(K) = \{A(K) : A \in \mathfrak{S}_n\}$, then $\mathfrak{S}_n(K)$ is dense in $\mathcal{K}_n$. It might be conjectured that for general $K \in \mathcal{K}_n$, $\mathfrak{S}_n(K)$ is dense in $\{C \in \mathcal{K}_n : \ \mathfrak{F}_n(C) \subseteq \mathfrak{F}_n(K)\}$.

3.4 Proof of Theorem 3.

In this subsection we show that the set of all K such that $\mathfrak{P}_n(K) = \mathbb{R}^n$, is dense in $\mathcal{K}_n$ and consequently the set of all K such that $\mathfrak{P}_n(K) = \mathfrak{F}_n(K)$ is dense in $\mathcal{K}_n$. A further corollary is that, for every $k \in \mathbb{N}$, $0 \le k \le n$, there exists a convex body Q_k such that $\mathfrak{P}(Q_k)$ is a k-dimensional affine subspace of $\mathbb{R}^n$.

It is relatively easy to construct examples of convex bodies K in the plane such that $\mathfrak{P}_n(K) = \mathbb{R}^2$. To do so in higher dimensions is more involved and we present a construction in the proof of Theorem 3 below. First, we will briefly mention two examples in the plane.

Example 1. Let S be a regular simplex in the plane and let $\mathcal{J}(S)$ be the ellipsoid of maximal area inscribed in S. We show in the section below that the center $j(S)$ of $\mathcal{J}(S)$

is an affine invariant point. We can assume that $\mathcal{J}(S) = B_2^2$, the Euclidean ball centered at 0 with radius 1. Then e.g. $S = \text{conv}\left((-1, -\sqrt{3}), (-1, \sqrt{3}), (2, 0)\right)$.

Let $0 < \lambda < 1$ be such that $H((1+\lambda)e_1, e_1) \cap \text{int}(S) \neq \emptyset$ and consider the convex body $S_1 = S \cap H^+((1+\lambda)e_1, e_1)$ obtained from S by cutting of a cap from S. Then still $j(S_1) = 0$ but the center of gravity has moved to the left of 0. Next, let $\gamma > 0$ be such that $H((1+\gamma)u, u) \cap \text{int}(S_1) \neq \emptyset$, where $u = \frac{(-1,\sqrt{3})}{2}$ and consider the convex body $S_2 = S_1 \cap H^+((1+\gamma u), u)$ obtained from S_1 by cutting of a cap from S_1. Then still $j(S_2) = 0$ but the center of gravity $g(S_2)$ of S_2 has moved and it is different from the Santaló point $s(S_2)$ of S_2. $j(S_2)$, $g(S_2)$ and $s(S_2)$ are three affinely independent points of $\mathbb{R}^2$, hence span $\mathbb{R}^2$.

Example 2. Let S be the equilateral triangle in the plane centered at 0 of Example 1 with vertices $a = (2, 0)$, $b = (-1, \sqrt{3})$ and $c = (-1, -\sqrt{3})$. Then, as noted in Example 1, B_2^2 is the John ellipse $\mathcal{J}(S)$ of S. Let b_1, c_1 be two points on the segments $[a, b]$ and $[a, c]$, such that the segment $[b_1, c_1]$ does not intersect B_2^2. Then B_2^2 is still the John ellipse of the quadrangle $\text{conv}\,(b, b_1, c_1, c)$. Now the Löwner ellipse $\mathcal{L}(T)$ of the triangle $T = \text{conv}\,(b_1, b, c)$ is centered at $\frac{1}{3}(b_1 + b + c) \neq 0$, if $b_1 \neq a$. $\mathcal{L}(T)$ intersects the segment $[a, c]$ at c and at some point c'. When $b_1 \to a$, one has $\mathcal{L}(T) \to 2B_2^2$ and thus $c' \to a$. So we may choose b_1 such that $[b_1, c']$ does not meet B_2^2, and thus for some $c'' \in [a, c]$, $[b_1, c_1]$ does not meet B_2^2 for any $c_1 \in [c', c'']$. Finally, let $P(c_1)$ be the quadrangle $P(c_1) = \text{conv}\,(b, b_1, c_1, c)$, with $c_1 \in [c', c'']$. Since b_1, b, c and c_1 are the vertices of $P(c_1)$ and $c_1 \in \mathcal{L}(T)$, $\mathcal{L}(T)$ is also the Löwner ellipsoid $\mathcal{L}(P(c_1))$of $P(c_1)$. Altogether,

The John ellipse of $P(c_1)$ is B_2^2 which is centered at 0, so that the affine invariant point $j(P(c_1)) = 0$.

The Löwner ellipse of $P(c_1)$ is centered at $\frac{1}{3}(b_1 + b + c)$, so that the affine invariant point $l((P(c_1)) = \frac{1}{3}(b_1 + b + c) \neq 0$.

An easy computation shows that the centroid of $P(c_1)$ moves on an hyperbola when c_1 varies in $[c', c'']$.

So, in general, these three points are not on line. $\square$

Proof of Theorem 3. The set of n-dimensional polytopes is dense in $(\mathcal{K}_n, d_H)$. Let P be a polytope and let $\eta > 0$ be given. Then it is enough to show that there exists a convex body Q with $d_H(P, Q) < \eta$ and such that $\mathfrak{P}_n(Q) = \mathbb{R}^n$.

We describe the idea of the proof. For a properly constructed convex body Q we will construct $\Delta_i \in \mathfrak{P}_n$, $1 \leq i \leq n+1$, in such a way that the $\Delta_i(Q)$ are affinely independent.

The construction of such a Q is done inductively: we first construct Q_1 very near P and such that $\Delta_1(Q_1)$ is near an extreme point v_1 of Q_1. Then we construct Q_2 very near Q_1 and P and such that $\Delta_1(Q_2)$ is near the extreme point v_1 of Q_2 and $\Delta_2(Q_2)$ is near an extreme point $v_2 \neq v_1$ of Q_2.

Let $P = \text{conv}\,(v_1, \dots, v_m)$ be a polytope with non-empty interior and with m vertices, $m \geq n+1$. We pick $n+1$ affinely independent vertices of P. We can assume that these are $v_1, \dots, v_{n+1}$. Let $0 < \eta_1 < \frac{\eta}{n+2}$ be given. By Lemma 5, there exists $z_1 \in P$, $\|v_1 - z_1\| \leq \eta_1$, and $0 < r_1 \leq \eta_1$ such that $B_2^n(z_1, r_1) \subseteq P$ and such that

$$Q_1 = \text{conv}\,(B_2^n(z_1, r_1), v_2, \dots, v_m)$$

has $v_2, \dots, v_m$ as extreme points,

$$d_H(Q_1, P) \leq \eta_1, \tag{23}$$

and for sufficiently small δ_1,

$$\|v_1 - g\left(Q_1 \setminus (Q_1)_{\delta_1}\right)\| \leq 2\eta_1. \tag{24}$$

We let $\varepsilon_1 < \eta_1$ and choose an ε_1-net $\mathcal{P}_{\varepsilon_1}$ on $\partial\left(B_2^n(z_1, r_1)\right)$ and put

$$P_1 = \operatorname{conv}\left(\mathcal{P}_{\varepsilon_1}, v_2, v_3, \dots, v_m\right).$$

Then $P_1 \subseteq Q_1 \subseteq P$ and $d_H(P_1, Q_1) \leq \varepsilon_1 < \eta_1$. By Corollary 1, for a given $K \in \mathcal{K}_n$, for a given $0 < \delta < \left(\frac{n}{n+1}\right)^n$ and $\varepsilon > 0$, there exists $\gamma(K, \delta, \varepsilon)$ such that if

$$d_H(K, L) < \gamma(K, \delta, \varepsilon), \quad \text{for } L \in \mathcal{K}_n, \quad \text{then} \quad \|g(K \setminus K_\delta) - g(L \setminus L_\delta)\| < \varepsilon. \tag{25}$$

As $d_H(P_1, Q_1) \leq \varepsilon_1$, we get that

$$\|g(Q_1 \setminus (Q_1)_{\delta_1}) - g(P_1 \setminus (P_1)_{\delta_1})\| < \eta_1,$$

if we choose in addition ε_1 such that $\varepsilon_1 < \gamma(Q_1, \delta_1, \eta_1)$. Thus, together with (24),

$$\|v_1 - g(P_1 \setminus (P_1)_{\delta_1})\| \leq 3\ \eta_1. \tag{26}$$

Please note that $v_2, \dots, v_m$ are extreme points of P_1. Now we apply Lemma 5 to P_1. Let $\eta_2 < \min\{\varepsilon_1, \gamma(P_1, \delta_1, \eta_1)\}$. By Lemma 5 there exists $z_2 \in P_1$, $\|v_2 - z_2\| \leq \eta_2$, and $0 < r_2 \leq \eta_2$ such that $B_2^n(z_2, r_2) \subset P_1$ and such that

$$Q_2 = \operatorname{conv}\left(\mathcal{P}_{\varepsilon_1}, B_2^n(z_2, r_2), v_3, \dots, v_m\right)$$

has $v_3, \dots, v_m$ as extreme points,

$$d_H(Q_2, P_1) \leq \eta_2, \tag{27}$$

and for sufficiently small δ_2,

$$\|v_2 - g\left(Q_2 \setminus (Q_2)_{\delta_2}\right)\| \leq 2\eta_2. \tag{28}$$

As $\|v_1 - z_1\| \leq \eta_1$ and $\|v_2 - z_2\| \leq \eta_2$, we have that $d_H(Q_2, P) \leq \eta_1$. Moreover, as $d_H(Q_2, P_1) \leq \eta_2 < \gamma(P_1, \delta_1, \eta_1)$, we get by (25) with $\varepsilon = \eta_1$ and by (26) that

$$\|v_1 - g(Q_2 \setminus (Q_2)_{\delta_1})\| \leq \|v_1 - g(P_1 \setminus (P_1)_{\delta_1})\| + \|g(P_1 \setminus (P_1)_{\delta_1}) - g(Q_2 \setminus (Q_2)_{\delta_1})\| \leq 4\eta_1.$$

Now we let $\varepsilon_2 < \min\{\eta_2, \gamma(Q_2, \delta_1, \eta_1)\}$, choose an ε_2-net $\mathcal{P}_{\varepsilon_2}$ on $\partial\left(B_2^n(z_2, r_2)\right)$ and put

$$P_2 = \operatorname{conv}\left(\mathcal{P}_{\varepsilon_1}, \mathcal{P}_{\varepsilon_2}, v_3, \dots, v_m\right).$$

Then $P_2 \subseteq Q_2 \subseteq P$ and $d_H(P_2, Q_2) \leq \varepsilon_2$. By (25), with $\varepsilon = \eta_2$, and if we choose in addition $\varepsilon_2 < \eta(Q_2, \delta_2, \eta_2)$, we get

$$\|g(Q_2 \setminus (Q_2)_{\delta_2}) - g(P_2 \setminus (P_2)_{\delta_2})\| < \eta_2$$

and thus, together with (28),

$$\|v_2 - g(P_2 \setminus (P_2)_{\delta_2})\| \leq 3\ \eta_2. \tag{29}$$

Please note that $v_3, \ldots, v_m$ are extreme points of P_2. Now we apply Lemma 5 to P_2. Let $\eta_3 < \min\{\varepsilon_2, \gamma(P_2, \delta_2, \eta_2)\}$. By Lemma 5 there exists $z_3 \in P_2$, $\|v_3 - z_3\| \le \eta_3$, and $0 < r_3 \le \eta_3$ such that $B_2^n(z_3, r_3) \subset P_2$ and such that

$$Q_3 = \mathrm{conv}\,(\mathcal{P}_{\varepsilon_1}, \mathcal{P}_{\varepsilon_1}, B_2^n(z_3, r_3), v_4, \ldots, v_m)$$

has $v_4, \ldots, v_m$ as extreme points,

$$d_H(Q_3, P_2) \le \eta_3, \tag{30}$$

and for sufficiently small δ_3,

$$\|v_3 - g\,(Q_3 \setminus (Q_3)_{\delta_3})\,\| \le 2\ \eta_3. \tag{31}$$

As $\|v_1 - z_1\| \le \eta_1$, $\|v_2 - z_2\| \le \eta_2$ and $\|v_3 - z_3\| \le \eta_3$ we have that $d_H(Q_3, P) \le \eta_1$. Moreover, as $d_H(Q_3, P_2) \le \eta_3 < \gamma(P_2, \delta_2, \eta_2)$, we get by (25) with $\varepsilon = \eta_2$ and (29) that

$$\|v_2 - g(Q_3 \setminus (Q_3)_{\delta_2})\| \le \|v_2 - g(P_2 \setminus (P_2)_{\delta_2})\| + \|g(P_2 \setminus (P_2)_{\delta_2}) - g(Q_3 \setminus (Q_3)_{\delta_2})\| \le 4\eta_2.$$

As $d_H(Q_2, Q_3) \le \varepsilon_2 < \gamma(Q_2, \delta_1, \eta_1)$, it follows from (25) with $\varepsilon = \eta_1$ that

$$\|g(Q_2 \setminus (Q_2)_{\delta_1}) - g(Q_3 \setminus (Q_3)_{\delta_1})\| \le \eta_1.$$

By (30), it also follows from (25) with $\varepsilon = \eta_1$ that

$$\|g(P_1 \setminus (P_1)_{\delta_1}) - g(Q_2 \setminus (Q_2)_{\delta_1})\| \le \eta_1.$$

This, together with (26) gives

$$\begin{aligned}\|v_1 - g(Q_3 \setminus (Q_3)_{\delta_1})\| &\le \|v_1 - g(P_1 \setminus (P_1)_{\delta_1})\| + \|g(P_1 \setminus (P_1)_{\delta_1}) - g(Q_2 \setminus (Q_2)_{\delta_1})\| \\ &\quad + \|g(Q_2 \setminus (Q_2)_{\delta_1}) - g(Q_3 \setminus (Q_3)_{\delta_1})\| \\ &\le 5\eta_1.\end{aligned}$$

We continue to obtain $Q = Q_{n+1}$ and affine invariant points $\Delta_i = g(Q \setminus Q_{\delta_i})$, $1 \le i \le n+1$, such that for all i,

$$\|v_i - \Delta_i(Q)\| \le (n+2)\eta_1 < \eta.$$

As for $1 \le i \le n+1$, the v_i are affinely independant, so are the Δ_i.

It remains to show that $\mathcal{O}_n = \{K \in \mathcal{K}_n : \mathfrak{P}_n(K) = \mathbb{R}^n\}$ is open in $(\mathcal{K}_n, d_H)$. Observe that $K \in \mathcal{O}_n$ if and only if for some $p_1 \ldots, p_{n+1} \in \mathfrak{P}_n$ (depending on K),

$$\mathrm{vol}_n(\mathrm{conv}\big(p_1(K) \ldots, p_{n+1}(K)\big) > 0.$$

Since $L \to \mathrm{vol}(\mathrm{conv}\big(p_1(L) \ldots, p_{n+1}(L)\big)$ is continuous on $\mathcal{K}_n$, it follows that $\mathcal{O}_n$ is open. $\square$

Corollary 2. *For every $k \in \mathbb{N}$, $0 \le k \le n$, there exists a convex body Q_k such that $\mathfrak{P}(Q_k)$ is a k-dimensional affine subspace of $\mathbb{R}^n$.*

Proof. For $k = 0$, we take a centrally symmetric body. For $k = n$, we take the body Q of Theorem 3. For $1 \le k \le n-1$, we take the intermediate bodies Q_k constructed in the proof of Theorem 3. $\square$

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