Exploring Categorical Structuralism

COLIN MCLARTY*

Hellman [2003] raises interesting challenges to categorical structuralism. He starts citing Awodey [1996] which, as Hellman sees, is not intended as a foundation for mathematics. It offers a structuralist framework which could defined in any of many different foundations. But Hellman says Awodey's work is 'naturally viewed in the context of Mac Lane's repeated claim that category theory provides an autonomous foundation for mathematics as an alternative to set theory' (p. 129). Most of Hellman's paper 'scrutinizes the formulation of category theory' specifically in 'its alleged role as providing a foundation' (p. 130). I will also focus on the foundational question. Page numbers in parentheses are page references to Hellman [2003].

1. Categorical Set Theory

Hellman correctly notes (p. 131) that Mac Lane often writes as if he used a set-theoretic foundation. Indeed Mac Lane does use a set-theoretic foundation—categorical set theory. Categorical set theory begins with the theory WPT (for *well-pointed topos*) which is the elementary topos axioms plus *extensionality*, also called *well-pointedness*:

> For any parallel functions $f, g\colon A \to B$, either $f = g$ or there is some $x \in A$ with $fx \neq gx$.

A function is fully determined by its values on elements.[1] An element is a function from a singleton set; so $x \in A$ says $x : 1 \to A$. In ZF each element $x \in A$ determines a unique function from any singleton 1 to A, taking the single element of 1 to x, and vice versa. In categorical set theory the function is the element.

* Department of Philosophy, Case Western Reserve University, Cleveland, Ohio 44106 U. S. A. cxm7@po.cwru.edu

[1] Extensionality makes the internal logic classical as noted in Mac Lane and Moerdijk [1992], p. 331. Hellman stresses that 'the first-order topos axioms must be supplemented so that Boolean, classical, logic will be available. (Without supplementation, which Mac Lane and Moerdijk [1992] specify in detail, the underlying logic of an elementary topos will be intuitionistic.)' (p. 131). It is a detailed matter in general topos theory. For categorical set theory the extensionality axiom is all the supplement you need.

Mac Lane's preferred foundation is 'a well pointed topos with Axiom of Choice and a Natural Number Object' which, he notes, are the axioms of Lawvere's *Elementary Theory of the Category of Sets*, or ETCS (Mac Lane [1998], p. 291; citing Lawvere [1964]). Hellman wants to explore whether, and how, this can serve as a foundation. He raises five main issues:

(1) He asks whether it is too weak, saying: 'a powerful set theory (although not all of ZFC) can be "recovered" in topos theoretic terms' (p. 131).

(2) Is it very complicated? As to how it handles the real numbers Hellman says, 'An example might be the construction of a ring of "real numbers" in a topos of sheaves based on a topological space' (p. 133).

(3) It presupposes too much. 'There is frank acknowledgement that the notion of *function* is presupposed, at least informally, in formulating category theory' (p. 133).

(4) The question of mathematical existence 'really just does not seem to have been addressed!' (p. 136).

(5) 'It lacks an external theory of relations' (p. 138).

As to the first, Hellman knows there are extensions of ETCS as strong as ZFC but he queries whether they are truly topos theoretic. He specifically questions the true toposness of the axiom scheme of replacement (p. 144) and large cardinal axioms (p. 139). Of course, categorical set theory is not supposed to be just topos theory. It is supposed to extend the general topos axioms by those particulars which distinguish the topos of sets from other toposes. These distinctions naturally include replacement and some aspects of cardinality. I discuss these in more technical detail at the end of this note.

2. Constructing the Reals

Hellman notes that you can translate ZF into ETCS terms (as we describe in Section 7), and so you can translate all of ZF mathematics into ETCS. While ETCS does not prove all the axioms of ZF in this form, it proves enough for most working mathematics. So he says 'one might expect Mac Lane to take advantage of this' by declaring math is founded in ETCS since it is founded in a fragment of ZF that translates to theorems of ETCS. But he also speculates 'Perhaps, however, Mac Lane has in mind an independent development of ordinary mathematics in topos theory which does not take a detour through set theory. An example might be the construction of a ring of "real numbers" in a topos of sheaves based on a topological space (as is done in Mac Lane and Moerdijk [1992], although even this mimics Dedekind's famous construction fairly closely)' (p. 133). There are several things to say about this.

First, there is no need to speculate. In Mac Lane [1986] he describes the independent development of ETCS for ten pages before the single paragraph on interpreting membership in ETCS (pp. 392–402). There the interpretation is used to compare categorical and membership-based foundations and hardly as a strategy for categorical foundations. The interpretation never appears at all in the discussion of foundations in Mac Lane [1998], which we already quoted urging ETCS.[2]

Second, ETCS is a set theory. It is not a membership-based set theory like ZF. It is a function-based set theory. Mac Lane generally uses the phrase 'set theory' to mean ZF, a habit of more than thirty years before ETCS was conceived. But we cannot let his terminology misdirect us. He is explicit that ETCS is his preferred account of sets. Where Hellman speaks of a detour through set theory, he could better speak of a detour through membership-based set theory. Of course I argue that categorical foundations make no such detours.

Third, the topos construction of the Dedekind reals does not mimic Dedekind's construction. It *is* Dedekind's construction. Admittedly Dedekind did not work in topos theory. Nor did he work in ZF, Zermelo's theory, or even in Cantor's set theory, none of which existed in 1858 when he produced his definition of the reals (Dedekind [1963]). He worked with sets and subsets and orderings, all of which can be formulated in Cantor's set theory, or in ZF, or in topos theory—the latter two achieving a level of formal rigor that Cantor did not.

The main philosophic point is that constructing the reals in a topos of sheaves on a topological space obviously presupposes topological spaces. Hellman observes what every category theorist knows when he says 'the notion of a topological space ... is taken for granted in the ordinary workings of category theory ... But the categorical foundationalist cannot take these notions for granted' (p. 136). So no one does.

ETCS foundations for the real numbers begin with the categorical axiom of infinity. It says there is a set supporting particularly simple recursive functions:

> There is a set $\mathbb{N}$ with an element $0 \in \mathbb{N}$ and a *successor* function $s\colon \mathbb{N} \to \mathbb{N}$ with this property: For any set T plus initial value $x \in T$ and function $f\colon T \to T$, there is a unique $u\colon \mathbb{N} \to T$ such that $u(0) = x$ and $us = fu$.

[2] Mac Lane lived in Hermann Weyl's house for some months in 1932–33 as a graduate student and worked with him on the philosophy of mathematics. Mac Lane's doctoral dissertation was on proof theory. He studied Carnap's philosophical logic and often discussed it with his friend Willard Quine. He was early on the Executive Committee of the Association of Symbolic Logic. With all this in mind, one might *not* expect him to make such a baroque and naïve false step as urging that a translation of ZF into ETCS makes ETCS a foundation for mathematics. But expectations apart, his publications are clear.

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "x"'] & N \arrow[r, "s"] \arrow[d, dashed, "u"] & N \arrow[d, dashed, "u"] \\ & B \arrow[r, "f"'] & B \end{tikzcd}$$

This is rather different from the ZF axiom of infinity and closer to Dedekind with his emphasis on functions. In ETCS, this implies the Peano axioms and gives the usual arithmetic of the natural numbers.

To construct the integers in categorical set theory, consider the subset

$$\{\langle m, n, m', n'\rangle \mid m + n' = m' + n\} \rightarrowtail (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$

This is the *equalizer* of two obvious functions from $(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$ to $\mathbb{N}$. Equalizers are sets of solutions to equations and the ETCS axioms say they exist. This one is an equivalence relation $\langle m, n\rangle \sim \langle m', n'\rangle$ on $\mathbb{N} \times \mathbb{N}$ by trivial arithmetic. So it has a *coequalizer* $q\colon \mathbb{N} \times \mathbb{N} \twoheadrightarrow \mathbb{Z}$ which we take as defining the set of equivalence classes $\mathbb{Z}$. The function q maps each pair $\langle m, n\rangle$ to the integer $m - n$.

Saying $q\colon \mathbb{N} \times \mathbb{N} \twoheadrightarrow \mathbb{Z}$ is coequalizer means that a function $f\colon \mathbb{Z} \to T$ to any set T just amounts to a function $\tilde{f}\colon \mathbb{N} \times \mathbb{N} \to T$ with the property that:

$$\langle m, n\rangle \sim \langle m', n'\rangle \text{ implies } \tilde{f}(m, n) = \tilde{f}(m', n').$$

When an integer $i \in \mathbb{Z}$ corresponds to $\langle m, n\rangle$ we have $f(i) = \tilde{f}(m, n)$. Then we define $+\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ and $\times\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ in the obvious way on pairs $\langle m, n\rangle$ and get integer arithmetic. And the natural numbers are included in the integers by taking each n to the pair $\langle n, 0\rangle$.

We construct the rational numbers as equivalence classes on $\mathbb{Z} \times \mathbb{N}$ for the equivalence relation

$$\langle i, n\rangle \sim \langle i', n'\rangle \text{ iff } s(n') \times i = s(n) \times i'.$$

That is, we treat $\langle i, n\rangle$ as the rational number $i/s(n)$ so denominators are positive natural numbers. Rational arithmetic derives from integer arithmetic by the obvious functions on pairs $\langle i, n\rangle$. The real numbers can be defined by Dedekind cuts, or Cauchy sequences. The two give the same result up to isomorphism.

In ZF a quotient of an equivalence relation on S is defined to be a subset of the powerset of S, then it provably has the coequalizer property, and all of this is uniquely defined in terms of elements. In ETCS the quotient is defined as a coequalizer, one way to prove it exists is to use the same subset of the powerset of S, and all this is defined only up to isomorphism. Quotients do all the same things on either definition. And ETCS defines powersets differently from ZF. Again ZF defines them uniquely by elements, and ETCS defines them up to isomorphism. But powersets do all the same things either way.

We have a long way to go before meeting sheaves on topological spaces. The foundational point is: Once you get beyond axiomatic basics, to the level of set theory that mathematicians normally use, ZF and ETCS are not merely inter-translatable. They work just alike.

3. Presupposition

Hellman follows Feferman [1977] saying categorical foundations presuppose a theory of sets and functions. More precisely: 'There is frank acknowledgement that the notion of *function* is presupposed, at least informally, in formulating category theory' (p. 133). What does it mean to presuppose something informally? For one thing it means the axioms do not formally presuppose any such thing. If they did, Feferman or Hellman could show where they do it. It turns out to mean the axioms are *motivated by* an informal idea of function.

Certainly ETCS is motivated by an informal idea of sets and functions. Even the general category axioms were motivated twenty years earlier by an informal idea of functions. Hellman (pp. 134–135) gives reasons why this is illegitimate, at least in the case of categorical axioms, which I cannot entirely understand. It may hang on his belief that 'category theory—at least as presented in axioms—is "formal" or "schematic": unlike the axioms of set theory, its axioms are not *assertory*'(p. 135). But this confuses the general category axioms with specific categorical foundations such as ETCS. I discuss this further in the next section.

4. Mathematical Existence

Hellman raises the question of mathematical existence. What mathematical objects are supposed to exist? He finds that categorists have never even tried to answer. That surprised me. So let me be clear that he does say it: On the problem of mathematical existence 'this problem as it confronts category theory can be put very simply: the question really just does not seem to be addressed!' (p. 136). He italicizes: '*What categories or topoi exist?*' and more formally '*what axioms govern the existence of categories or topoi*?' He finds the categorists leave us 'in the situation of the Walrus and the Carpenter, after the oysters were gone: 'but answer came there none" ' (p. 137).

A footnote says Lawvere [1966] gave an answer on the category of categories as a foundation (CCAF) but Isbell [1967] suspected it was inconsistent. In fact Lawvere suggested several successively stronger answers. Isbell thought one might be overly strong, inconsistent, and he suggested a repair. A search of 'category of categories' on *Mathematical Reviews* turns up Blanc and Preller [1975], Gray [1969], McLarty [1991], and Tsukada [1981] further pursuing Lawvere's ideas. And that search misses others I know. Anyone who wants to explore CCAF will have oysters around for a while yet.

More importantly, each proposed categorical foundation is a proposed answer. Categorical set theory says: the sets and functions posited by the axioms exist (and thus the categories constructed from those sets exist). Synthetic differential geometry (SDG), taken as a foundation, says: the smooth spaces and maps posited by the axioms exist (and thus also the categories constructed from them).

So Hellman's claim that 'category theory ... *lacks substantive axioms for mathematical existence*' is a misunderstanding (p. 138, Hellman's italics). Indeed category theory *per se* has no such axioms, but that is no lack, since category theory *per se* is a general theory applicable to many structures. Each specific categorical foundation offers various quite strong existence axioms.

Nothing in ZF foundations prepares us for such variety. Anyone who speaks of 'set axioms' more or less has to mean axioms for some potentially foundational theory such as ZF. There is little other use for axioms on sets.[3] In contrast the 'category axioms', as given by Eilenberg and Mac Lane [1945], are used in myriad ways to myriad ends in the daily practice of mathematics. These are the axioms Mac Lane and others insist have no one intended interpretation. Hellman quotes Awodey [1996] to this effect on p. 134. These axioms make no existence claims and they are constantly used in many different interpretations. Only they are not specifically the ETCS axioms, or the CCAF axioms.

So Hellman says set theorists believe their axioms describe a 'set-theoretic reality', and then says: 'Surely category theorists do not intend their first-order axioms to be read in any such fashion. It is not as if the category theorist thinks there is a specific "real world topos" that is being described by these axioms!' (p. 137).

If he means the category axioms *per se*, this is true but irrelevant. No one proposes those as a foundation. If he means the ETCS axioms, or the CCAF axioms, or axioms for SDG, the claim is relevant but false. As to particulars, though, the category theorist has different ideas, depending on which category theorist it is.

Mac Lane ([1986], p. 441) says 'the real world is understood in terms of many different mathematical forms'. He adopts Popperian empirical falsi-

[3] There are some parallels in the ZF tradition. Barwise in such works as Barwise and Etchemendy [1987] suggests that variant set theories, of the kind studied in Aczel [1988], might each be suited to analysis of different philosophic problems. But this remains a more or less foundational use. And the range of variant axioms for membership-based set theory is nothing compared to the range of different categories.

fiablity as the criterion of scientific truth and accordingly says statements of mathematics cannot be true since they are deduced from axioms ([1986], pp. 91 and 440 ff.). He finds 'Mathematical existence is not real existence' (p. 443) where real existence is understood in physicalist empirical terms. Similarly, I doubt that any ZF set theorist takes set-theoretic reality as empirical or physical reality in anything like a Popperian way. (Maddy [1990] goes the farthest of anyone I know to make sets physical or empirical, but she is no Popperian.)

In Mac Lane's words, with his emphasis, 'Mathematics *is* a formal network, but the concepts and axioms there are based on 'ideas'. What are they? ... They are not like Platonic ideal forms because they are intuitive and vague' ([1986], p. 444). He suggests many philosophic questions about these 'ideas' and gives some thoughts without trying to settle on answers. Graphics throughout his [1986] depict networks of relations among parts of mathematics.

These ideas are real for Mac Lane though they are not empirical or physical reality. For Mac Lane the ideas of mathematics as a whole, which are themselves constantly developing and extending, are the prior reality to any foundation. He views foundations as 'proposals for the organization of mathematics' ([1986], p. 406).

Most to our point here, Mac Lane notes the organizing role of our ideas on sets. We can formalize these in ZF set theory, but he prefers foundations in ETCS, 'axiomatizing not elements of sets but functions between sets' ([1986], p. 398). Throughout his work Mac Lane takes it that we know of 'the category of sets, as formulated in the Zermelo-Fraenkel axioms' and we can also formulate that same category by ETCS (Mac Lane and Moerdijk [1992], p. 331). The category is the prior reality and both formulations describe it though they say somewhat different things about it. He always rests his mathematics on 'the usual category of all sets' and advocates formalizing it by ETCS ([1998], pp. 290 and 291).

Lawvere's first publication on ETCS noted that 'there is essentially only one category which satisfies the eight axioms together with the (nonelementary) axiom of completeness, namely, the category $\mathcal{S}$ of sets and mappings' (Lawvere [1964], p. 1506). He took that category as a reality, which we must describe, fully aware that no first-order axioms can completely describe it. Unlike Mac Lane, Lawvere believes mathematical statements can be true. He refers to the past 150 years of mathematics to say 'from the ongoing investigation of the ideas of sets and mappings, one can derive a few statements called *axioms*; experience has shown that these statements are sufficient for deriving most other true statements' on sets and functions (Lawvere [2003]). By 'true' statements he means true of real sets and functions.

Lawvere differs from Mac Lane in not preferring ETCS as the one foundation for mathematics. He and Mac Lane both take the mathematical ideas of the category of categories, and the category of smooth spaces, as further realities, though initially vague and intuitive enough. But Mac Lane would formulate them all within categorical set theory. Lawvere has also formulated these and other ideas in their own terms. He conceives axioms on the category of categories as an alternative foundation to ETCS. He first aimed to axiomatize general topos theory as a framework in which to axiomatize the category of smooth spaces (see his 1967 lectures published as Lawvere [1979]).

Hellman rightly sees the general topos axioms are understood as general and not assertory. But their extension to axioms for Synthetic Differential Geometry (SDG) is asserted as describing a particular mathematical reality. Those axioms are asserted of the category of smooth spaces.

Bell ([1986], [1988]) deals with 'local set theory' as an approach to topos theory. Again, Hellman correctly sees this as a general account, not assertory. But the point of the general account is that a specific local set theory may be asserted as a foundation for some specific topic or purpose. As an example Bell [1998] gives the most extensive account to date of how SDG describes the mathematical reality of shapes and motions in the world.

He gives quick infinitesimal proofs of numerous classical theorems in geometry, mechanics, hydrodynamics, and thermodynamics. On the most immediate interpretation, *much* of SDG agrees with differential geometry in ZFC, but SDG also has infinitesimals and infinitesimal spaces. Bell quotes numerous philosophers and mathematicians, from Aristotle to René Thom, to argue that SDG describes the actual continuum better than the standard set-theoretic real line does ([1998], pp. 1–12). He says of infinitesimals: 'Physicists and engineers, for example, never abandoned their use as a heuristic device for deriving (correct!) results in the application of the calculus to physical problems' ([1998], p. 3, his parenthesis and exclamation mark). SDG is about actual shapes and motions in the world—not shaped and moving physical objects *per se*—but the mathematical reality of shapes and motions as Aristotle discusses that distinction in his *Physics* II.2 ([1984], pp. 330ff.).

5. Actual and Possible

Hellman also wants to consider possibilities in foundations and he assimilates categorical foundations to this in an interesting way (pp. 145 ff.). But there is a fundamental difference. Hellman wants one theory of all the possibilities. Mac Lane and Lawvere say many theories could possibly be used and indeed should be used. Lawvere does not say ETCS, and CCAF, and SDG are all possible. He says the category of sets, and the category of categories, and the topos of smooth spaces, are all actual, and so are many others yet unthought of, which we should aim to think of. When Mac Lane urges against fixed foundations, his concern is explicitly that 'any such fixed foundation would preclude the novelty which might result from the discovery of new form' ([1986], p. 455). This holds just as true for any fixed conception of possibilities as for any fixed conception of actualities.

Hellman says 'whatever the representational powers of a well-pointed topos they are ultimately limited and readily transcended' (p. 145). This is true. The WPT axioms without replacement are transcended even by the cardinal $\aleph_\omega$. But Mac Lane has been clear on many occasions that the easy kinds of transcendence do not interest him. See notably his debate with Adrian Mathias in Mathias [1992], Mac Lane [1992], Mathias [2000], Mac Lane [2000]. He happily foregoes $\aleph_\omega$ as unused in mainstream mathematics. What interests him are new forms that could fundamentally change our daily approach to mathematics, fundamentally new mathematical ideas, as have arisen in the past, and undoubtedly will in the future. Talk of possibility *versus* actuality does not bear very much on the creation of this kind of novelty.

Hellman's ideas on how to 'make modal-logical sense of functorial relations among categories', where the categories are taken as merely possible, are interesting (p. 147). I take them as progress towards a kind of categorical nominalism.

6. External Relations

Category theory also '*lacks an external theory of relations*' according to Hellman (p. 138), following Feferman. This is true if, as Feferman maintains, an external theory of relations must be supplied by his projected '*general* theory of collections and operations' (Hellman, p. 134, discussing Feferman [1977]). Feferman has never yet given his general theory and this probably explains why 'Mac Lane never responded directly to Feferman's critique' (p. 133).

Actually Hellman seems to mean something less exigent by an external theory of relations than Feferman does. Hellman finds that ZF already provides such a theory. But he says categorical foundations do not: 'Precisely because the "axioms" for categories and topoi are to be understood structurally' and thus they assert nothing, so that 'we must appeal to some prior, external, assumptions to prove any theorems *about* these structures' (p. 135n). This is quite true of the category axioms *per se*, and of the general topos axioms. That is why no one offers them as foundations for mathematics. The ETCS axioms, on the other hand, or the CCAF axioms, or the axioms of SDG, are assertory in the same way as the ZF axioms. We have covered this.

7. Inter-Interpretation

The obvious interpretation of ETCS into ZF interprets sets as sets, functions as functions, composition as composition. The ETCS axioms other than choice become theorems of ZF and indeed theorems of the much weaker theory BZ using bounded separation (any subclass of a set, defined by a formula with only bounded quantifiers, is a set). The ETCS axioms altogether become theorems of BZC.

In fact, the axioms of ETCS become familiar truths about sets. The axioms of extensionality and of infinity, above, are examples. Another would be the product set axiom:

> For any sets A and B there is a set $A \times B$ with projection functions $p_1 \colon A \times B \to A$ and $p_2 \colon A \times B \to B$ such that, for any set T and functions $f \colon T \to A$ and $g : T \to B$ there is a unique $\langle f, g\rangle : T \to A \times B$ with $p_1\langle f, g\rangle = f$ and $p_2\langle f, g\rangle = g$.

$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[d, dashed, "<f{,}g>"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

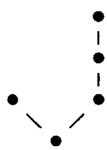
So there is no risk that categorical set theory uses odd ideas. It uses ideas we all do use. The arguable risk is, that it is too abstract. One could argue that it elides the membership-based grounding we need really to understand these ideas. I reply that virtually no one except logicians ever learns this membership-theoretic grounding so it cannot possibly be requisite to mathematical understanding.

Going the other way there are two important interpretations of ZF in categorical set theory. The *homophonic* one is partial. You interpret sets as sets, functions as functions, composition as composition. You do *not* interpret membership between sets, $A \in B$. So this interpretation is not defined for most axioms of ZF, but it is for many theorems. Technically, take the standard definitional extension of ZF by terms for function, domain, codomain, and composition. The sentences of this extension which do not contain '$\in$', and where no variable treated as a function is also treated as a domain or codomain, have homophonic translation to ETCS.

For example there is no homophonic interpretation of the ZF empty-set axiom: There is a set $\emptyset$ such that no set A has $A \in \emptyset$. This relies directly on membership of sets. But consider this ZF theorem: There is a set $\emptyset$ such that for every set A there is exactly one function $\emptyset \to A$. This is homophonically interpretable, and is also a theorem of ETCS. Indeed all isomorphism-invariant theorems of ZF have homophonic interpretation. The ones mathematicians normally use are all theorems of ETCS. Ones dealing with $\aleph_\omega$ or higher cardinals require extensions of ETCS.[4]

[4] Mathias ([2001], §9) proves a sharp result: While ETCS proves the existence of each

There is a tree-theoretic total interpretation of ZF in ETCS with Mac Lane and Moerdijk [1992] as one good reference. Each ZF set has a membership tree. For example $S = \{\emptyset, \{\{\emptyset\}\}\}$ has this tree:



The bottom node has two nodes above it, meaning S has two elements. The left-hand node has none above it, meaning one element of S is the empty set. The right-hand node codes a singleton set whose sole element is empty. It is not terribly hard to say which branching patterns represent ZF sets. I will call them simply *trees*.

Categorical set theory has no membership relation between sets but it can describe trees. We can interpret ZF sets as trees, and interpret the ZF relation $A \in B$ as:

> A tree representing A occurs on some node immediately above the root of any tree representing B.

Then every ZF statement appears as an ETCS statement about trees. The axioms of ZFC other than separation or replacement are all theorems of ETCS. So are all bounded instances of separation. In short, the axioms of BZC become ETCS theorems.

So the ETCS axioms and the BZC axioms are inter-interpretable. Each language can be translated into the other, so that the axioms of one become theorems of the other. This implies that any extension of one (in the same language) is inter-interpretable with some extension of the other (in the same language). Of course proof theory cannot show that every plausible, natural extension of one theory corresponds to a plausible, natural extension of the other. 'Plausible' and 'natural' are not terms of proof theory.

8. Replacement in Categorical Set Theory

Cantor says 'Two equivalent multiplicities are either both "sets" or both inconsistent' *i.e.*, they are both proper classses ([1967], p. 114). In slightly more modern terminology: Any multiplicity in bijection with a set is a set. The idea is somehow implicit in much of his earlier work, as he very freely forms larger and larger cardinals with no explicit justification, including cases that today are most naturally justified by the axiom scheme of replacement.

'beth n', $\beth_n$ the n-th iterated powerset of ω, for natural numbers n, it cannot prove: for every $n \in \mathbf{N}$ there exists a sequence $\beth_0, \beth_1, \ldots, \beth_{n-1}$ of the first n beths.

A moment's reflection shows that equivalence, or bijection, is an anodine complication. Cantor's assertion amounts to: every multiplicity which is the image of some set under some function, is a set. The serious foundational question is what are a 'multiplicity' and a 'function'? We want a first-order formal theory of sets. So we take a 'function' to be a functional relation stateable in our theory and its range of values on any set is a 'multiplicity' which this axiom declares to be a set.

There is a technical point. The theory ETCS is structural in the sense that each ETCS set provably has all the same properties as any set isomorphic to it. An ETCS formula can only specify a set up to isomorphism.[5] So let us introduce a quantifier $(\exists!_i S)$, read 'there exists a set S, unique up to isomorphism'. Formally

$$(\exists!_i S)\ P(S) \quad \text{abbreviates} \quad (\exists S)(\forall X)(\ P(X) \Leftrightarrow X \cong S\ ).$$

Our axiom scheme of replacement says:

> For each relation $R(x, Y)$ of arrows x to sets Y in ETCS: For every set A, if $(\forall x \in A)(\exists!_i S_x)\ R(x, S_x)$ then there is some $f : S \to A$ such that for each $x \in A$ the set S_x is the inverse image of x along f.

It is easier to see in a diagram. Suppose each $x \in A$ is assigned a set S_x, unique up to isomorphism, by a relation $R(x, Y)$ expressible in ETCS. Then there is a set S and arrow $f : S \to A$ with this property: for each $x \in A$ and any S_x with $R(x, S_x)$ there is some $S_x \to S$ making this a pullback:

$$\begin{tikzcd} S_x \arrow[r, tail] \arrow[d] & S \arrow[d, "f"] \\ 1 \arrow[r, "x"'] & A \end{tikzcd}$$

So S is a disjoint union of the S_x and f gives the structure of a set of sets $\{S_x |\ x \in A\}$ as each S_x is the preimage of its x. The theory ETCS+R gotten by adding this axiom scheme to ETCS is inter-interpretable with ZFC. The next section gives a proof.

Probably few readers think this looks like the ZF axiom scheme. That scheme has no homophonic translation into ETCS as it uses membership so heavily. It has a tree-theoretic translation, since every ZF formula does, but you can see our axiom does not deal in trees. Our axiom is not a translation from ZF. It is a plain categorical version of Cantor's idea.

[5] For a detailed statement and proof see McLarty [1993]. We could change that by a *skeletal axiom* saying any two isomorphic sets are equal. Then specification up to isomorphism is unique specification. I avoid that here, to avoid any misimpression that the skeletal axiom is doing serious work. I feel skeletal axioms never do serious work, but other people may disagree.

Hellman finds the motivation for the axiom scheme 'remains characteristically set theoretic' and is 'to guarantee that the domain of sets and functions is incomparably more vast' than what follows in BZC or ETCS by themselves (p. 144). Indeed. Categorical set theory is a theory of sets. Its motives are set theoretic.

We can ask: are there motives for the axiom scheme of replacement apart from the ZF version of the scheme, and more generally apart from Zermelo's form of membership-based set theory? But that is obvious. Cantor did not have Zermelo's membership-based conception. Zermelo criticized him sharply over it (notably at Cantor [1932], pp. 351, 353). And replacement is Cantor's idea. Zermelo missed it in his axioms until Fraenkel and Skolem pointed it out fourteen years later (see Maddy [1988], pp. 489 f.). Cantor's conception and not Zermelo's motivated the axiom. Hellman rightly suggests we have no 'new motivation' for the axiom (p. 144). Neither did Fraenkel or Skolem. We all have Cantor's original motive which predates the membership-based ZF version by decades.

That is all the pedigree the scheme needs to be honest categorical set theory. Yet the scheme also has a topos-theoretic motive. Topos theory arose as a theory of pasting and coverings—in general that means coverings of a topological space by open subsets, or an arithmetic scheme by *étale* neighborhoods, and so on. The above axiom scheme of replacement says precisely that the discrete points of a set A form a cover: Given a set S_x over each point x of a set A, you can paste together a set of sets $f\colon S \to A$.

This is radically false in a general topos since it treats each object A as a discrete collection of points. And that is precisely the distinction of the topos of sets. Sets *are* discrete collections of points as expressed by the *well-pointedness* axiom. From a topos viewpoint the axiom scheme powerfully expresses the discrete nature of sets.

9. Proof

To prove ETCS+R is inter-interpretable with ZFC we first prove our replacement scheme gives separation with unbounded quantifiers. Then we prove a technical result replacing arbitrary relations by relations functional up to isomorphism:

Theorem 1. *For any ETCS formula $G(x)$ with x a variable over arrows, ETCS+R proves every set A has a subset $S \rightarrowtail A$ such that for all $x \in A$, $x \in S$ if and only if $G(x)$.*

Proof. From $G(x)$ define a relation $R(x, Y)$:

$$(\ G(x) \text{ and } Y \cong 1\ ) \quad \text{or} \quad (\ \text{not-}G(x) \text{ and } Y \cong \emptyset\ ).$$

Each fiber is a singleton or empty. So the arrow $f : S \to A$ given by the replacement axiom is monic, a subset. To say the fiber S_x is non-empty is the same as to say $x \in S$. And so $x \in S$ if and only if $G(x)$. □

Theorem 2. *For any ETCS formula $F(x,Y)$ with x variable over arrows, and Y over sets, there is a formula $F_m(x,Y)$ such that ETCS proves*

$$(\forall A)(x \in A)\ [(\exists C)F(x,C)\ \Leftrightarrow\ (\exists!_i C)F_m(x,C)].$$

Proof. By the axiom of choice, all sets are comparable in cardinality and every set of cardinality classes is well ordered. By unbounded separation, given x, Y with $F(x,Y)$, the power set of Y includes the set of all subsets Y' of Y such that $F(x,Y')$. The minimal elements of that set are unique up to isomorphism and agree up to isomorphism with the minimal elements gotten from any other set Z with $F(x,Z)$. So let $F_m(x,Y)$ say: $F(x,Y)$, and Y has a one-to-one function to every Z with $F(x,Z)$. □

Now we show that in ETCS our replacement scheme implies the one in Osius ([1974], p. 116).[6] By using powersets Osius's version is equivalent to saying: For every ETCS formula $F(x,Y)$

$$\forall P \exists B \forall (x \in P)[(\exists C)F(x,C) \Rightarrow (\exists C)(C \leq B\ \wedge\ F(x,C))].$$

Here $C \leq B$ says there is a one-to-one function from C to B. But this follows from the instance of our replacement scheme where $R(x,Y)$ is:

$$F_m(x,Y) \quad \vee \quad (Y \cong \emptyset\ \wedge\ \neg\exists Z\ F(x,Z)).$$

That is, $\emptyset$ has a one-to-one function to every set, and the inverse image of any $x \in A$ along any $f: S \to A$ has a one-to-one function to S. So let $B = S$. Clearly the instances of our scheme all translate to theorems of ZFC. But Osius [1974] shows all instances of the ZF axiom scheme translate to theorems using his categorical scheme which follows from ours.

10. Large Cardinals

Hellman asks 'How can topos theory accommodate strong axioms of infinity, and can it do so "autonomously"?'(p. 139). The first question has a familiar answer: Large-cardinal axioms are usually given, by ZF set theorists, in

[6] Hellman says of Osius [1974]: 'In the case of the Axiom of Replacement one simply rewrites it in category theoretic notation' (p. 144). But Osius [1974] actually gives a reflection principle rather than replacement, and only uses the name 'axiom scheme of replacement' because it accomplishes the same thing. The principle he gives is descended from Lawvere's preprint [1963], shortened to appear as [1964]. Lawvere tells me that at the time he was discussing reflection principles with logicians at Berkeley. This one is not close in phrasing (let alone a mechanical translation) to any version I could find in set-theory textbooks but no doubt it reflects conversations with ZF set theorists.

isomorphism-invariant form. They are homophonically ETCS axioms in the first place.[7] So the answer to the second question is yes.

For example, a measurable cardinal is an uncountable cardinal κ with a non-principal κ-complete ultrafilter. In ETCS we interpret 'cardinal' as 'set' and give the exact same definition. The definition deals with sizes of sets, and with the elements of a set S being members of some subsets of S and not others. It does not use global membership (the idea that for every two sets A and B it makes sense to ask whether or not $A \in B$). It does not use well-founding. It uses no feature of ZF that is lacking in ETCS, with one exception. The ZF definition, as given, refers to cardinals as a distinguished kind of set while ETCS marks no such distinction. But the distinction plays no role even in ZF. It makes only a trivial difference, in ZF, if you replace 'cardinal' by 'set'.

At the technical heart of it: an ultrafilter $\mathfrak{U}$ on any set S is κ-complete if the intersection of every set of less than κ members of $\mathfrak{U}$ is a member of $\mathfrak{U}$. This deals with the size of κ but in no way asks what the elements of κ are. It never asks whether κ is a von Neumann ordinal, or any other encoding of ordinal. It uses no ordering on κ. It makes sense for any set κ. And it all lives inside the double power set of S. It asks which elements of S lie in which subsets, and which subsets of S lie in which sets of subsets. It never asks about membership of arbitrary sets in other sets.

Large cardinals are routinely pursued in isomorphism-invariant terms. The work, its motives, and whatever axioms it suggests, all translate homophonically into ETCS.

[7] Indeed one of Friedman's devices in his recent work towards new axioms of infinity, is precisely trees. See his 'Finite trees and the necessary use of large cardinals' (March 22, 1998), available as of this writing as a downloadable manuscript on his website at www.math.ohio-state.edu/foundations/. He regards trees as useful combinatory objects for motivating new axioms, as he also regards various other structures. This makes perfect sense from the categorical viewpoint as well. The categorical set theorist can look at all of Friedman's relational structures and the cardinality requirements they pose. The only difference is categorical set theorists will not ask which relational structures are realized by the 'actual membership relation' between sets, since categorical set theory has no membership relation between sets. It has membership only between elements and subsets of a given set.

References

ACZEL, PETER [1988]: *Non-well-founded Sets.* Stanford, California: Center for the Study of Language and Information.

ARISTOTLE [1984]: *The Complete Works of Aristotle.* Princeton: Princeton University Press.

AWODEY, S. [1996]: 'Structuralism in mathematics and logic', *Philosophia Mathematica* (3) **4**, 209–237.

BARWISE, JON, and JOHN ETCHEMENDY [1987]: *The Liar : An Essay on Truth and Circularity.* Oxford: Oxford University Press.

BELL, J. [1986]: 'From absolute to local mathematics', *Synthese* **69**, 409–426.

——— [1988]: *Toposes and Local Set Theories: An Introduction.* Oxford: Oxford University Press.

——— [1998]: *A Primer Of Infinitesimal Analysis.* Cambridge: Cambridge University Press.

BLANC, G., and A. PRELLER [1975]: 'Lawvere's basic theory of the category of categories', *Journal of Symbolic Logic* **40**, 14–18.

CANTOR, G. [1932]: *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts.* Berlin: Julius Springer.

——— [1899]: Letter to Dedekind 28 July 1899, in J. van Heijenoort, ed., *From Frege to Gödel.* Cambridge, Mass.: Harvard University Press, 1967, pp. 113–117.

DEDEKIND, R. [1963]: *Essays on the Theory of Numbers.* New York: Dover Publications.

EILENBERG, S., and S. MAC LANE [1945]: 'General theory of natural equivalences', *Transactions of the American Mathematical Society* **58**, 231–294.

FEFERMAN, S. [1977]: 'Categorical foundations and foundations of category theory', in R. Butts and J. Hintikka, eds., *Logic, Foundations of Mathematics, and Computability Theory.* Dordrecht: Reidel, pp. 149–169.

GRAY, J. [1969]: 'The categorical comprehension scheme', in Peter Hilton, ed. *Category Theory, Homology Theory and their Applications.* Berlin: Springer Verlag, pp. 242–312.

HELLMAN, G. [2003]: 'Does category theory provide a framework for mathematical structuralism?', *Philosophia Mathematica* (3) **11**, 129–157.

ISBELL, J. [1967]: Review of Lawvere [1966]. *Mathematical Reviews* **34**, no. 7332.

JUST, W., H. JUDAH, and H. WOODIN, eds. [1992]: *Set Theory of the Continuum.* Berlin: Springer Verlag.

LAWVERE, F. W. [1963]: 'An elementary theory of the category of sets', Lecture notes of the Deparment of Mathematics, University of Chicago.

——— [1964]: 'An elementary theory of the category of sets', *Proceedings of the National Academy of Science of the U. S. A.* **52**, 1506–1511.

——— [1966]: 'The category of categories as a foundation for mathematics', in S. Eilenberg *et al.*, eds., *Proceedings of the Conference on Categorical Algebra, La Jolla, 1965.* Berlin: Springer Verlag, pp. 1–21.

——— [1979]: 'Categorical dynamics', in *Topos Theoretic Methods in Geometry*, No. 30 in *Various Publications.* Aarhus: Aarhus University, pp. 1–28.

——— [2003]: *Sets for Mathematics.* Cambridge: Cambridge University Press.

MAC LANE, SAUNDERS [1986]: *Mathematics: Form and Function*. Berlin: Springer Verlag.

________ [1992]: 'Is Mathias an ontologist?', in Just, Judah, and Woodin [1992], pp. 119–122.

________ [1998]: *Categories for the Working Mathematician*. 2nd ed. Berlin: Springer Verlag.

________ [2000]: 'Contrary statements about mathematics', *Bulletin of the London Mathmatical Society* **32**, 527.

MAC LANE, SAUNDERS, and I. MOERDIJK [1992]: *Sheaves in Geometry and Logic*. Berlin: Springer Verlag.

MADDY, PENELOPE [1988]: 'Believing the axioms', *Journal of Symbolic Logic* **53**, 481–511 and 736–764.

________ [1990]: *Realism in Mathematics*. Oxford: Oxford University Press.

MATHIAS, A. R. D. [1992]: 'What is Mac Lane missing?', in Just, Judah, and Woodin [1992], pp. 113–118.

________ [2000]: 'Strong statements of analysis', *Bulletin of the London Mathematical Society* **32**, 513–526.

________ [2001]: 'The strength of Mac Lane set theory', *Annals of Pure and Applied Logic* **110**, 107–234.

MCLARTY, C. [1991]: 'Axiomatizing a category of categories', *Journal of Symbolic Logic* **56**, 1243–1260.

________ [1993]: 'Numbers can be just what they have to', *Noûs* **27**, 487–498.

OSIUS, G. [1974]: 'Categorical set theory: A characterization of the category of sets', *Journal of Pure and Applied Algebra* **4**, 79–119.

TSUKADA, H. [1981]: 'Category theory not based upon set theory', *Sci. Papers College Gen. Ed. Univ. Tokyo* **31**, 1–24.

ABSTRACT. Some issues for categorical foundations for mathematics are explored. Among them: how categorical foundations assert the existence of their objects and arrows, and how categorical set theory motivates its own axiom scheme of replacement. Two interpretations of ZF into categorical set theory are explicitly compared. One is homophonic but partial; the other is total and uses membership trees.