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‘Mathematical Platonism’ *Versus* Gathering the Dead: What Socrates teaches Glaucon†

Colin McLarty*

Glaucon in Plato’s *Republic* fails to grasp intermediates. He confuses pursuing a goal with achieving it, and so he adopts ‘mathematical platonism’. He says mathematical objects are eternal. Socrates urges a seriously debatable, and seriously defensible, alternative centered on the destruction of hypotheses. He offers his version of geometry and astronomy as refuting the charge that he impiously ‘ponders things up in the sky and investigates things under the earth and makes the weaker argument the stronger’. We relate his account briefly to mathematical developments by Plato’s associates Theaetetus and Eudoxus, and then to the past 200 years’ developments in geometry.

> Plato was much less prodigal of affirmation about metaphysical ultimates than interpreters who take his myths literally have supposed. (Paul Shorey [1935], p. 130)

> Mathematics views its most cherished answers only as springboards to deeper questions. (Barry Mazur [2003], p. 225)

Glaucon in Plato’s *Republic* always tries to agree with Socrates but it is not easy for him. He has trouble with intermediates (μεταξύ). He is like Socrates speaking with the wise woman Diotima in Plato’s *Symposium* (202–204). At first Socrates thought every being was either mortal or immortal, wise or ignorant, and so on. Diotima taught him of intermediates (again, μεταξύ) such as daimons between mortal and immortal and especially the search for wisdom between ignorance and being wise (204a).

† All Greek is translated from the Perseus website www.perseus.tufts.edu consulting the translations listed below. Quotes from Plato’s *Republic* were checked in S. R. Slings *Platonis Rempublicam* (Oxford, 2003). Plato is cited by Stephanus page numbers, and Aristotle by Bekker page numbers, found in the margins of most editions.

Martha Nussbaum and Stanley Rosen, by writings and conversation, taught me to read Plato. Debra Nails offered many improvements to an earlier draft of this paper. A version given to the Philosophy Colloquium at the University of Notre Dame benefitted from discussion by Patricia Blanchette, Mic Detlefsen, Lynn Joy, Alasdair MacIntyre, Kenneth Sayre, and others. Bill Tait and anonymous referees gave valuable arguments on mathematics and on Plato. John Mayberry especially insisted on a clear interpretation of Aristotle on intermediates. Of course none can be supposed to endorse my conclusions.

* Department of Philosophy, Case Western Reserve University, Cleveland, Ohio 44106 U. S. A. cxm7@po.cwru.edu

He accepted each one but repeatedly failed to find them himself. Glaucon is worse. He accepts the intermediates Socrates offers and then forgets them.

So Glaucon mistakes compulsion toward a goal for achievement of it, and adopts 'mathematical platonism' in today's sense. Paul Bernays coined the term with a less specific meaning than it has today. He used it to describe David Hilbert's view of mathematics as 'cut off from all links with the reflecting subject' and his sole reference to Plato is: 'this tendency asserted itself especially in the philosophy of Plato' ([1983], p. 258). Today mathematical platonism says: 'mathematics is the study of distinctively mathematical objects which exist independently of us . . . [with] no particular position in space or time. They are immutable and eternal' (Blanchette [1998], p. 120). Philosophers of mathematics differ on relating this to Plato.[1]

In the dialogues only Glaucon takes this view. He says 'Geometry is knowledge of what always is' (*Republic* 527b). Socrates first ignores his claim, then gives increasingly sharp corrections, and eventually laughs at him. Socrates' account leads away from mathematical platonism via the protean notion of ἀναιροῦσα, discussed below, into gathering dead bodies and a lottery in heaven. Other dialogues, notably the *Meno*, talk of mathematics without addressing this issue.

This paper shows how Socrates urges a specific alternative to platonism; and, what is more, a good one. Glaucon alternates between the extremes of platonism and empiricism.[2] Socrates makes the mathematical subjects 'mere preludes to the song itself that we must learn' (531d). Mathematical objects summon us towards the forms without existing as forms. The song is dialectic which will accept no conclusions until it grasps the good and can derive all from that. In particular, *contra* Tait [2002], dialectic is no program for formalizing mathematics, politics, or any other science. Neither is it poetic excess. It is an account of intellectual life bearing more on goals than achievements, and less on textbook mathematics than on research. It is a valuable perspective on the actual practice of mathematics.

We look briefly at Plato's associates Theaetetus and Eudoxus this way. But little is known of them. The value of Socrates' song, lotteries, and gathering the dead shows better today in the profusion of geometries since Gauss and Riemann.

[1] Blanchette [1998] does not discuss Plato. Balaguer ([1998], p. 185, n12) and Maddy ([1990], p. 20) note the theme is platonic. Mayberry ([2000], pp. 14 ff.) denies the view is Plato's but argues for a specific relation to Plato. Brown ([1999], p. 9), Hersh ([1997], p. 95), Shapiro ([2000], p. 53), and Tait [2002] claim it is Plato's own belief.

[2] Annas well says 'In the dialogues, then, there is no explicit commitment to a platonist theory of numbers, but there is evidence that Plato found it natural to think of numbers platonistically [in today's sense]' (Annas [1999], p. 5). At *Republic* 526a, quoted below, it is natural to Glaucon while it is 'risky' to Socrates.

1. Plato in General

A popular argument says mathematical platonism obviously follows from Plato’s theory of forms. But it cannot be obvious. On one hand many Plato scholars who disagree on many other things agree that Plato gives no theory of forms. Rosen says ‘there is no theory of a uniform set of beings, usually called “Ideas” or “forms”, in the Platonic dialogues’ ([1987], p. 337). Williams says Plato’s dialogues offer ‘no Theory of Forms’, ‘there is really no such thing’ ([1999], pp. 9, 32). Plato believes a philosopher must long to know the forms, but his character Socrates consistently says he does not know them.

Cooper notes the dialogues criticize a series of theories of forms. The longest discussion of them, in the *Parmenides*, criticizes many in turn and at best points the way towards more. Cooper argues that the *Parmenides* starts with forms as described in Plato’s *Symposium*, *Phaedo*, and *Republic*, and shows ‘Socrates’ efforts to articulate a theory of forms have been premature’ (Cooper [1997], p. 359). Socrates repeatedly insists his account will be premature, if indeed any true account is possible (for the *Republic*, see especially 506b–507a, and 533a).

Nor is it obvious that Plato took mathematical objects to have the character of forms. Burnyeat says ‘for a Platonist the Forms are yet more real and still more fundamental to explaining the scheme of things than the objects of mathematics’ ([2000], p. 22, *cf.* 34 ff.). Mathematical objects are not perfectly real. Burnyeat makes this gap central to Plato’s view of mathematics as leading beyond itself towards the good.

As Plato left no obvious theory of forms, let alone one that obviously includes mathematical objects, we must look to the details. We especially notice which character makes which claims. So this paper does not directly confront Annas’s claim about Plato’s implicit views: ‘It is clear from the dialogues, though nowhere explicitly stated in them, that Plato is a platonist in the sense current in the philosophy of mathematics’ ([1999], p. 3). We look to the explicit. Glaucon affirms platonism while Socrates offers a brilliant alternative. Contrary to Annas, I incline to believe that here Socrates states Plato’s views. But I will not argue for that. Either way, Socrates’ position is worth considering.[3]

2. *Republic V–VI*

The ontology of *Republic V* opposes ‘what is’ to ‘what is not’ and contrasts both to an intermediate. Socrates links this to knowledge, ignorance, and opinion: ‘Knowledge is over what is, while ignorance is necessarily over what is not’ (477a). ‘Opinion is intermediate between those two’ and is

[3] Nails [1999] shows how little is achieved by claiming a character speaks for Plato.

over the ‘intermediate between what purely is and what in every way is not’ (478d).[4] These intermediates are physical objects. For example, each physical object is big (compared to some things) and is not big (compared to others). More vividly, as opposed to the one beautiful-itself or the just-itself:

> of the many beautiful things will any not also appear ugly? Is one of the just things not [in some respect] unjust? Is one of the pious things not impious? (479a)

Book *VI* refines this. It takes up knowing and being as in book *V* except that it omits ignorance and ‘what is not’. Socrates says:

> When [the soul] focuses where truth shines and on what is, it understands, knows, and appears to have understanding, but when it focuses on what is mingled with obscurity, on what comes to be and passes away, it opines and is dim sighted. (508d)

He fills this out with his simile of the divided line.

He divides the kinds of things into visible and intelligible (νοητός) realms each with two sections, the way one could divide a line in two and then divide the divisions. The first section of the visible contains shadows and reflections. These appear as images of the physical objects in the second visible section. Physical objects in turn appear as images in the first intelligible section. That section contains things which ‘the soul, using as images the things which were imitated before, is compelled to study by hypotheses, but moving to a conclusion rather than to a first principle’ (510b). These include the objects of geometry with visible, physical, shapes as images of them.

Shapiro overstates, saying ‘Plato’s world of geometry is divorced from the physical world and, more important, geometrical knowledge is divorced from sensory observation’ ([2000], p. 55). Geometry is only seeking a divorce. Geometers take hypotheses from the visible and proceed by reason but with no reasoned critique: ‘they make these their hypotheses and find no value in giving an account of them, to themselves or to others, as if they were clear to everyone’ (510c). Socrates says: ‘They use visible forms and talk about them, though they are not thinking of these but of those things of which they are a likeness . . . seeking to see things which can be seen no way other than in thought’ (510de).

The final section escapes the senses to find ‘an unhypothetical first principle—from hypotheses but without the images of the other section, making a method for itself from the forms themselves’ (510b). It starts from ‘genuine hypotheses, or stepping stones’ and rises beyond them

[4] Translations in this paper use ‘intermediate’ always and only for μεταξύ. The word ‘knowledge’ translates always and only γνῶσις. Similarly ‘thought’ is for διάνοια; cognates of ‘understanding’ for cognates of νόος; ‘science’ for ἐπιστήμη.

to 'the unhypothetical first principle of everything' before descending to conclusions (511b). This describes dialectic.

Glaucon says he grasps this 'not adequately' and he summarizes. Those who study 'the so-called crafts, which start from hypotheses':

> are compelled to use thought and not their senses, yet because their investigation rests on hypotheses rather than a first principle, they seem to you not to understand the objects, although these could be understood from a first principle. You seem to me to call the state of mind of the geometers thought but not understanding, as thought is intermediate between opinion and understanding (511cd).

Socrates replies 'Your exposition is most adequate' and repeats with approval Glaucon's names for the sections.

Socrates does not repeat Glaucon's claim that the objects of the hypothesis-based crafts could be understood from a first principle. This claim amounts to mathematical platonism. Or, more precisely, it would amount to mathematical platonism if the first principle lay within mathematics. But we learn nothing about it. Socrates passes it by. When Socrates gives a view, Glaucon restates it, and Socrates affirms the restatement, Socrates must mean it. A view Socrates does not give, and does not affirm nor even examine when Glaucon offers it, he apparently does not hold.

Socrates and Glaucon agree the state of the geometers is intermediate between understanding and opinion. Thus the geometers' objects are doubly intermediate. They lie between 'what is' and the visible things, while visible things lie between what purely is and what in every way is not.[5] In book *VII* Socrates will revise some aspects of this, and especially complain that his use of 'science' has been too loose and 'based on habit' (533d).[6] But he will reaffirm that mathematics needs 'some other name connoting more clarity than opinion, and less than science' (533d) and he will again call it 'thought'.

Glaucon quickly forgets. On one hand he is spirited the way a philosopher must be. He is the 'example of what erotic men do' admiring all boys and not only some of them, so that he is like a philosopher who desires all wisdom and not only part of it (474d, recalled on 485a). But spirit and good memory are rarely combined (490c, 491b). Glaucon has less of the other more careful virtues of a philosopher than his brother Adeimantus, and less than Aristotle.

[5] Curiously, Georg Cantor compared transfinite sets to another double intermediate: 'what Plato in his dialogue "Philebus or the highest Good" calls μικτόν' (Cantor [1932], p. 204). Cantor notes this μικτόν is a mix of the finite and the infinite (*e.g., Philebus* 25b). The finite is between (μέσα) unity and the infinite (17a).

[6] He early used science for all crafts (*e.g.*, 340e, 342c), the science of boxing (422c), the science of making houses (438c), among others.

3. Aristotle on Plato's Mathematicals

According to Aristotle, Plato 'states that besides sensible things and the forms there exist intermediate things, the objects of mathematics, which differ from the sensible by being ἀΐδια and ἀκίνητα' (*Metaphysics* 987b, *cf.* 995b and 997b). The words ἀΐδια and ἀκίνητα are often translated 'eternal and immutable'. Thus the eternal and immutable objects in today's mathematical platonism. But that is not Plato's usage.[7] Plato uses ἀΐδιος for the stars at *Timaeus* 40b right after describing their creation in time. He uses it for souls (*Republic* 611b, *Phaedo* 106d) and the many gods (*Timaeus* 37c). These are all temporal, though enduring. They are not 'what always is.'

More pointedly, Socrates says geometers keep their hypotheses fixed (ἀκίνητος) and this blocks them from the clear vision of dialectic (*Republic* 533c). This gives textual support to the claim that Plato called mathematical objects ἀκίνητος. But it says mathematics is an obstacle to knowing what is. The word occurs just one other time in the *Republic*: for prisoners with their heads fixed in place (ἀκίνητος) in the cave (515a). Fixity, in the *Republic*, is the literal and the figurative paradigm of how we fail to know what is.

For Plato to call mathematical objects enduring (ἀΐδια) and fixed (ἀκίνητα) would mean they are more real than what comes to be and passes away, but less than what always is. Annas ([1999], p. 20) among others questions Aristotle's reliability here. So far as he is reliable, though, he confirms that Plato was not a mathematical platonist.

4. Pritchard and Tait against Mathematical Intermediates

There is an argument to show Plato rejected mathematical intermediates, since *Republic V* says different faculties (δύναμις, a power or capacity) are over different objects. Socrates and Glaucon agree opinion and science are different faculties. Socrates says:

> if knowledge is over what is, and ignorance necessarily over what is not, then for what is over the intermediate we must seek some intermediate between ignorance and science . . .. opinion is over one thing and science another since opinion and science are different faculties. (477ab)
>
> . . . for a faculty I look to just one thing—what it is over and what it accomplishes. This is why I call each itself a faculty, and that which is over the same thing and accomplishes the same thing I call the same faculty, and that which is over another and accomplishes another I call other. (477d)

[7] Throughout the *Politics* Aristotle uses ἀΐδιος to mean 'life-long', *e.g.*, 1271a.

A faculty or power is whatever is *over* (ἐπί) any thing, and is identified by the thing is it over.[8]

Tait says 'The faculties explicitly mentioned there are opinion and knowledge .... I don't see that there is room for intermediates'([2002], p. 15). He argues that Plato's exact sciences deal with forms. Pritchard also finds just two faculties but makes the opposite conclusion: 'Neither Plato nor Aristotle is committed to an ontology of separately existing mathematical objects' ([1995], p. 111). He argues that understanding for Plato deals with visible objects and Plato's mathematicians deal with visible diagrams and numbers ([1995], esp. pp. 92–97, 111).

On their face, though, the quotations deal with three faculties. Ignorance is the faculty over what is not. Indeed, if ignorance were not a faculty then opinion intermediate between it and knowledge would not be entirely a faculty either. Socrates does find this awkward. The problem of error and 'what is not' troubles him often.[9] But throughout this passage he continues to set ignorance over what is not. He never simply says what a modern philosopher might, and what Greek grammar certainly could express, that ignorance is not over things. Instead, with his divided line, he avoids 'ignorance' and 'what is not' by positing four states of the soul—imagination, belief, thought, and understanding—over four corresponding degrees of 'what is'.[10] Belief (taking the place of opinion) is over visible objects, understanding is over what is, so thought corresponds to intermediates between those.

The passage on 477–478 also calls sight and hearing faculties.[11] The lines on knowledge, ignorance, and opinion will not bear the weight of a final doctrine on faculties and being, much less of a final doctrine where only knowledge and opinion are faculties.

Tait argues that Plato took dialectic as a search for foundations of sciences such as geometry and a science of politics (Tait [2002], esp. pp. 18, 25). He well emphasizes that dialectic is not only moral and abstractly ontological. It also clarifies the sciences. But Plato saw it in a more living, chaotic sense, more related to research in geometry, for example, than to fixing a foundation for it. Socrates calls dialectic not a path to the mathematical sciences but a song to learn after them (531d).

[8] Accomplishment is not a further requirement. No faculty of the soul 'accomplishes' (ἀπεργάζεται) any specific thing in the *Republic* unless ἀπεργαστικὸν φιλοσόφου διανοίας at 527b is read to say geometry is 'an accomplishment of philosophic thought'.

[9] See Shorey [1935], note on 478b. Shorey must be right that Plato was not himself puzzled by this. Socrates labors the point to set the stage for a specific solution.

[10] Each state of the soul is 'over' its objects (511e); so each is a faculty.

[11] Sight is over light and hearing over speech (507–508). Courage is a faculty of the soul (429b, 430b) as is justice (433, 443b).

5. *Republic VII*

Plato's fullest discussion of mathematics is in *Republic VII* on educating guardians for the beautiful city. The education must suit soldiers and make them philosophers so it must draw their souls away from becoming and towards what is (521d and *passim*). The first subject Socrates finds is arithmetic, studied not as craftsmen and retailers do but in the sophisticated way of people clever in the subject. That means the 'ones' are simply ones, with no differences between them, and not attached to anything visible or tangible. Socrates asks:

> What do you suppose Glaucon? If someone asked them: 'O wondrous people (Ὦ θαυμάσιοι)[12] of what kind of numbers do you speak, in which the one is as you deem it, each one equal to every other, without the least difference and with no parts?' how would they answer?
>
> According to myself, they would answer that they were talking about those that can be grasped only in thought and can't be handled any other way.
>
> Then do you see, friend, I said, this subject risks being really compulsory to us, since it apparently compels the soul to consult understanding itself on the truth itself?
>
> Yes indeed, he said, it certainly does that. (526a)

Glaucon speaks enthusiastically of certainty and of himself. Socrates speaks dryly of wonders, likelihood, and appearance. He never agrees with Glaucon or the wondrous people on the being of numbers.[13] Rather, he finds arithmetic likely directs the soul towards understanding.

Socrates' term for risk (κινδυνεύει) refers to danger or daring in Plato and in ancient Greek generally. It is Polemarchus's answer in *Republic I* when Socrates traps him in a hopeless error:

> (Socrates) Then it is when money is useless that justice is most useful for it?
> (Polemarchus) It risks so. (333cd)

It is Thrasymachus's grudging answer when Socrates forces him to admit justice is like wisdom (350c). Through the *Republic* it occurs often. Sometimes it points to a false conclusion, sometimes true, but always to a problematic one. As to ancient Greek usage in general, Smyth lists it

[12] This common expression in the *Republic* is not a compliment. For example Socrates addresses Thrasymachus this way at 337b and 351e.

[13] Socrates gives a calmer view at *Philebus* 56de. The many can count and add two herds of cattle. The philosopher will not agree 'unless each of the many units is posited as different in no way from the others'. So philosophers can posit cattle as mere units. These units are ordinary things thought of in a philosophical way.

among the 'verbs and expressions of caution' ([1984], art. 2224a). Socrates is raising doubt against Glaucon's certainty.

The contrast grows stronger around geometry, which Socrates says will suit the guardians if it compels the soul towards being (526e). He said it does this in the simile of the divided line. But it also compels another way. Socrates recounts the exchange with Glaucon:

> Now no one, I said, with even a little experience of geometry will dispute that this science is entirely the opposite of what is said about it in the accounts of its practitioners.
> How is that? he said.
> They speak quite laughably, and that by compulsion, as if they are acting and all their accounts are for the sake of action. They talk of 'squaring', 'applying', 'adding' and the like, whereas the entire subject is pursued for the sake of knowledge.
> Absolutely, he said.
> And mustn't we agree further?
> To what?
> For knowledge of what always is,[14] and not what comes to be and passes away.
> That's easy to agree to, he said: Geometry is knowledge of what always is.
> It is a powerful device then,[15] my well-born friend, pulling the soul towards truth and accomplishing philosophic thought by directing upwards what we now wrongly direct downwards. (527ab)

Socrates corrects Glaucon. He states as a goal what Glaucon takes as achieved. He changes 'knowledge' to 'thought'. Both steps follow the agreement at 511d which Glaucon has forgotten.[16]

[14] Shorey [1935] misrenders 'Ὡς τοῦ ἀεὶ ὄντος γνώσεως as a sentence: 'It is the knowledge of that which always is'. The Greek includes no pronoun nor any verb. It is an adverbial phrase refining Socrates' claim: Geometry is pursued for the sake of knowledge 'as knowledge of the always being'. Here ὡς indicates purpose without indicating success or failure, *cf.* Smyth ([1984], art. 2992, 2996). See other textual evidence in Burnyeat ([2000], p. 39).

[15] Socrates draws an inference without endorsing Glaucon's statement. The inferential 'then' here is ἄρα: 'ἄρα marks a consequence drawn from the connection of thought, and expresses impression or feeling; the stronger οὖν marks a consequence drawn from facts' (Smyth [1984], art. 2787a). Socrates generally uses οὖν within his own account and ἄρα for conclusions he attributes to Glaucon.

[16] Mueller ([1992], p. 191) and Vlastos ([1973], p. 107n) note Glaucon and not Socrates says geometry is knowledge of what always is. They and Shapiro ([2000], p. 56) cite this exchange to show Plato believed mathematical objects are eternal.

Plato could write a plain declarative sentence. He did for Glaucon: 'Geometry is knowledge of what always is'.[17] But Socrates consistently says geometry is *for the sake of* knowledge, *for the sake of* knowing what always is. It is a *powerful device*[18] *pulling the soul towards* truth. Geometry produces philosophic thought, not knowledge. Socrates' words constantly match the terms of the divided line.

The dialogue goes on to show how unreliable Glaucon is on the nature and value of mathematics. At the same time it shows Socrates seeks no less laughable geometry. He lays out a huge program to make other sciences more like geometry, and suggests no improvement to geometry except better organized work on solids (528).

6. Glaucon and Further Mathematics

After praising geometry as knowledge of what always is, Glaucon praises the next subject, astronomy, as temporal: 'That's fine with me, for a better awareness of the seasons, months, and years is no less appropriate for a general than for a farmer or navigator.' Socrates' patronizing reply is literally 'you are sweet' (ἡδὺς εἶ). Reeve aptly translates 'you amuse me' (527d). This would be an insult if Glaucon were a grown man—and he more or less is.[19]

The hostile Thrasymachus taunted Socrates this way at 337d and 348c. Plato's audience would easily associate it with Glaucon since another version with the same literal meaning was γλυκὺς εἶ using 'glucus' close to Glaucon's name. Socrates uses that at *Greater Hippias* 288b. No young 'Glaucon' could escape being called 'sweet', though the name has finer associations as well.[20]

Glaucon tries again: 'since you rebuked me for praising astronomy in a vulgar way, I'll now praise it in your way, for I take it as clear to everyone that astronomy compels the soul to look upward' (528e). Wrong again. Socrates says astronomy as practiced at the time is worse than geometry. It actually reasons from visible things and not from hypotheses. Socrates makes a cutting joke.

Athenians knew the burlesque of Socrates, swinging about in a basket hanging from a crane, claiming to 'walk on air and contemplate the sun' in Aristophanes' comedy *Clouds* (line 223). Socrates quotes this and

17 Τοῦ γὰρ ἀεὶ ὄντος ἡ γεομετρικὴ γνῶσις ἐστιν.

18 specifically a ὁλκός, a machine for hauling large ships onto land.

19 Whatever the historical facts, the key point within the dialogue is that Glaucon and Adeimantus have distinguished themselves in battle outside Attica (368a). Citizens fought outside Attica after age twenty. See Nails ([2002], pp. 154–155).

20 For one, it suggests 'bright eyed', an Homeric epithet of Athena.

blames Aristophanes for the charges against him in Plato’s *Apology* (19c). In the *Republic* he turns a similar image against Glaucon: ‘your conception of “higher studies” seems quite generous, for if someone were to study something leaning his head back to see decorations on a ceiling, you would suppose he is studying not with his eyes but with his understanding’ (529ab). Glaucon is cast as a domesticated version of a famous and dangerous parody of Socrates.

Socrates in Plato’s *Apology* defends himself against what he calls the first false charge against him although it is not the official charge in the trial: ‘There is a wise man Socrates, who ponders things up in the sky and investigates things under the earth and who makes the weaker argument the stronger’ (18bc). He says the jury has heard: ‘Socrates does wrong and is a busybody, investigating things under the earth and in the heavens and making the weaker argument stronger’ and he names Aristophanes (19b). He repeats the charge at 23d. It was in Aristophanes’ *Clouds* twenty-four years before Socrates’ trial (vividly in lines 95–120 and obscenely at 190–195).

The character Socrates refutes Aristophanes’ charge point by point as he advises Glaucon: ‘let us approach astronomy through problems, as we do geometry, and leave things in the heavens alone, if we want to make the wise nature of the soul useful instead of useless’ (*Republic* 530bc). Our astronomy will not ponder things in the sky, just as geometry (earth measuring) does not study the earth, and so we make reason useful and not abusive.[21]

The last mathematical subject is harmonics. Socrates complains it is too concerned with the audible. Glaucon tries to agree by ridiculing musicians who argue over the smallest difference between distinct notes. Socrates goes along for a while, but says he had other offenders in mind. Even Pythagoreans who look for the numbers that produce harmonies do not look upwards. ‘Just like the astronomers, they seek the numbers in heard harmonies, not ascending to problems’. They do not rise to the level of geometry. Glaucon says ‘you speak of a daimonic task’ and Socrates replies ‘Yet it is useful in the search for the beautiful and the good’ (531c). Burnyeat [2000] gives a masterly interpretation of how this daimonic task helps us search for the good, and how the challenge affected Greek mathematics after Plato.

The reformed astronomy and harmonics aim for the level of geometry. There is no talk of raising them higher, nor of raising geometry. The only higher level mentioned in any Platonic dialogue is dialectic reaching the good or an unhypothetical first principle of everything.

[21] Netz ([1999], pp. 287, 300) shows how strongly ‘geometry’ kept the sense of land survey. For astronomy in the *Republic* see Burnyeat ([2000], p. 12).

7. *Meno*

Plato's *Meno* discusses how mathematics is known and not the being of mathematical objects. It is important for the vast and difficult question of exactly what hypotheses are for Plato.[22] Sometimes an hypothesis in a dialogue is a division into kinds, as geometers 'hypothesize' the three kinds of angle: acute, right, and obtuse (*Republic* 510c). Often it is an assertion someone will defend (*e.g. Theaetetus* 183b, *Sophist* 244c). Sometimes it is a first step meant to be surpassed by further inquiry (*Phaedo*, esp. 101 ff.). In the *Meno* Socrates uses it the way he says geometers often do, to mean an auxiliary concept useful in approaching a problem (86–87). As an 'hypothesis' to help find whether virtue can be taught, he introduces 'knowledge' and compares knowledge to both virtue and the teachable. In general an hypothesis in Plato is a context for an inquiry.[23]

Socrates in the *Meno* likens correct opinion to science: 'in no way is correct opinion inferior to science or less useful for action' (98c). He argues for this at length but draws no ontology from it. If he did draw an ontology from this epistemology, the way he did in the *Republic*, he would have to say: As science is like correct opinion, so the objects of science are like the objects of correct opinion. Those are visible objects. It would be a mistake to go farther and claim mathematical objects in the *Meno* are visible. In the same way, it is a mistake to claim mathematical objects in the *Republic* exist as what always is.

8. Two-fold Compulsion

Geometry compels the soul towards being and compels its practitioners to speak laughably. Socrates says both, at *Republic* 526–527, using the verb and the adjective from the same root, ἀνάγκη, meaning compulsion, force, or necessity. Geometers do not speak this way carelessly or by mistake. They do it by necessity, just as they necessarily turn towards what is. Throughout *Republic VI* and *VII* Socrates uses these words in the colloquial Greek senses of prison, or testimony under torture, as when he complains of Glaucon forcing him to answer, and in images of prisoners in the cave.

It is reasonably clear what ridiculous things geometers say. Socrates says they speak 'as if they are acting' (527a). Socrates also says why they are compelled. He is quoted above from 510bc introducing the first division

[22] See Mueller [1992], pp. 177–183 and Netz [forthcoming].

[23] The word ὑπόθεσις appears in Thomas [1998] only in Plato and Aristotle. It occurs just twice in Euclid's *Elements*. Positing unequal lines AC and DB, Euclid writes 'let AC be greater by hypothesis' for 'call the greater line AC' (*X*.44 and *X*.47). Thomas [1998] and Heath [1956] use 'hypothesize' for the Greek ὑπόκειμαι.

of the intelligible. Geometers use hypotheses taken from the visible, and do not give accounts of them. He repeats it at 511a:

> This is the sort I described as intelligible, though the soul is compelled to employ hypotheses in the investigation of it, not a first principle, because of its inability to extricate itself from and step above its hypotheses; it uses as images the very things of which images were made below.

Burnyeat ([2000], pp. 38 ff.) shows this is compulsory in Euclid’s practice.

Today the compulsion is beautifully seen in Hilbert’s *Foundations of Geometry*. At face value that is quite an active book about dropping lines, drawing lines, constructing angles, and it takes its axioms from the visible world. Hilbert says axiomatizing geometry is ‘equivalent to the logical analysis of our perception of space’ ([1971], p. 2). But then, Hilbert promoted neo-Kantian philosophy (see Peckhaus [1990], [1994]). Perhaps it is logical analysis, not of an empirical, but of a transcendental perception? Hilbert gives no such account.

On the other hand Hilbert famously said we need not take the language at face value. In a correct axiomatic treatment, he said, it must be possible to read ‘table, chair, beer stein’, in place of ‘point, line, plane’, and the proofs must still follow from the axioms (Blumenthal [1935], p. 403). They follow formally without regard to content. From this formal viewpoint the axioms are neither true nor false. There is no account to give of them. We can only draw consequences from them—as Socrates laughed at geometers for doing (510c and 533b *inter alia*).

Plato’s Socrates descried a real problem. We cannot rehearse the often brilliant and often vexed debates on empiricism and formalism in geometry. If one position in that debate entirely convinces you then you may find Plato’s views of purely historical interest. Only note that empirical geometry cannot satisfy Socrates, nor can formal geometry stripped of meaning and truth. Syntheses of these are more easily sought than found today as in Plato’s time.

Socrates says the philosopher’s goal is not to learn mathematics but to learn what mathematics is, and ‘if all our investigation of these studies brings out their community with each other and common origin and implies they are kin then our hard work has advanced our goal and is no useless labor, if not, it is useless’ (531cd). He finds their common origin is the twofold compulsion imposed by reasoning from the visible world: Reasoning compels our souls upwards, while dependence on the visible world makes that laughable. Arithmetic and geometry have risen farther. Astronomy and harmonics can rise to their level. None will criticize their own hypotheses. So Socrates says they are ‘mere preludes to the song itself that we must learn. For you surely do not imagine that people clever in these things are dialecticians’ (531d).

A typical wordplay contrasts mathematics and dialectic. Mathematics ‘compels’ the soul upwards, ἀναγκάζει. Dialectic never compels but ‘leads upwards’, ἀνάγει (533d and *passim*).[24] Plato plays with homophony most effusively in the *Symposium* but in all the dialogues to echo points his characters also make overtly. Mathematics and dialectic alike raise the soul. They differ crucially in how.

9. Socrates on Mathematics and Dialectic

Socrates says geometers ‘to some extent’ grasp what is and ‘they dream about what is, unable to get a waking view of it as long as they use hypotheses that they leave unaltered and that they cannot give an account of’ (533b). He earlier said ‘dreaming is mistaking what resembles a thing for that thing’ (476c). The objects of geometry resemble what is. He says only dialectic can achieve knowledge, which it does by ‘refuting (ἀναιροῦσα) hypotheses and proceeding to the first principle itself’ (533c).

Here ‘refute’ is the verb ἀναιρέω.[25] The Liddell and Scott *Intermediate Lexicon* gives many meanings for the word in Plato’s time, associating two especially to Plato. These include:

> 1. to take up, raise, 2. to take up and carry off, bear away, 3. to take up bodies for burial. II. to make away with, to destroy, kill. 2. of things, to abolish, annul. 3. to destroy an argument, confute (Plato). III. to appoint an oracle. . . . to take up for oneself, to gain, win, achieve. . . . 4. to take up new-born children. 5. to conceive in the womb. . . . 2. to undertake, contract for the execution of a work (Plato). . . . to take back to oneself, cancel.

Plato uses it just four other times in the *Republic*, all in the myth of how Er saw the afterlife without dying. Twice at 614b it means to gather corpses. Twice at 617e it means to pick up a numbered tile in a lottery. Taking those meanings, plus the two Liddell and Scott associate with Plato, dialectic proceeds by ‘gathering dead hypotheses’ and ‘winning hypotheses as prizes’, by ‘destroying hypotheses’ and ‘contracting to work out hypotheses’ until a first principle is reached.

Mueller reads the passage more gently. He translates it as ‘destroying hypotheses’, but says ‘the only destruction Socrates has in mind is the destruction of the hypothetical character of mathematical hypotheses under an unhypothetical starting point’ ([1992], p. 188). The hypotheses will not be changed but demonstrated from some first principle. It would be very odd, though, for Plato to use ἀναιροῦσα to mean essentially ‘proving’.

[24] It is immaterial whether these words have an actual etymological relation. This is play. *Cf.* Socrates’ whimsical account of ἀνάγκη at *Cratylus* (420d).

[25] This and Hegel’s ‘aufheben’ are both ‘up + lift’. Hegel could have named his dialectical step for Plato’s.

None of Plato’s dialogues leave their hypotheses unscathed, unless it would be the *Timaeus* or *Laws* which some people read as propounding theses with little questioning at all. If the dialogues are at least rough models of dialectic then dialectic will refute many hypotheses in the plainest way and all of them in some way before reaching the unhypothetical. Even if the *Timaeus* or *Laws* is properly read as unquestioning, neither one generates a first principle dialectically. No dialogue questions hypotheses and makes them unhypothetical.

Compare Socrates earlier saying dialectic treats hypotheses correctly not as first principles but ‘as stepping stones to take off from’, literally ‘as places to step and as *hormē* (ὁρμή)’ (511b). Liddell and Scott *Intermediate Lexicon* defines *hormē* in Plato’s time as:

> [I] a violent movement onwards, an attack . . . (examples) the rage of fire, the shock of a wave, the reach of a spear, [II] the first stir or start in a thing, an effort or attempt to reach a thing, impulse to do it, eager desire . . .

Plato uses this noun just four other times in the *Republic*. At 439b it names the impulse of a thirsty soul which ‘drives it like a wild beast to drink’. At 451c it refers to the start in life the city gives its warrior guardians. At 506e defining the good is beyond Socrates’ *hormē* or power. At 611e it is the soul’s yearning to be divine and free of this earth. This is not merely to secure our current beliefs. It is a wild urge towards what we do not have.

10. Geometry Then and Lately

Mueller contrasts ‘the generally smooth working of mathematics with the rough-and-tumble of the Socratic examination of doctrines’ ([1992], p. 180). But textbook mathematics is one thing and research another. Euclid’s *Elements* is a textbook. Almost certainly much of it was research in Plato’s life. Ancient and modern historians generally accept that Theaetetus and Eudoxus were friends of Plato and made major contributions on proportions, irrationals, and solid geometry, which became the culmination of Euclid’s *Elements*.[26]

Plato’s *Theaetetus* has Socrates talk with Theaetetus on the day Theaetetus gave the first general proof about irrationals. That is unlikely to be historical but Theaetetus may well have proved what became *Elements X*.9: If an integer is not the square of an integer, then its square root is irrational. Plato’s *Timaeus* uses the classification of regular solids generally credited to Theaetetus. Socrates at *Republic* 528 complains that cities

[26] See Theaetetus in Caveing ([1998], esp. pp. 302–309). Burnyeat ([2000], p. 63) suggests Eudoxus may have been too young to influence Plato’s *Republic*.

do not support solid geometry and he says it needs a commander to direct research.

Neither irrationals nor solid geometry were well organized even a century later in Euclid's *Elements*. Book *X* on irrationals is painful. Every other book of the *Elements* has a dramatic ending: the Pythagorean theorem, finding mean proportionals, finding all even perfect numbers, *etc*. Book *X* hardly ends at all. The longest book by far, it has myriad definitions, no apparent organization, and experts are not sure which was its last theorem (Heath [1956], vol. 3, p. 255). Solid geometry fills books *XI–XIII*, the last generally accepted books of the *Elements*. They are beautiful geometry but only vaguely axiomatized. They rely on proof by exhaustion which is generally credited to Eudoxus. It was controversial enough that Archimedes (or perhaps a scholiast) cites Eudoxus and Euclid as precedents to justify using it (Heath [1956], vol. 3, p. 65).

Probably these subjects developed a great deal during Plato's life (perhaps 427–347 B. C.) and from then until Euclid (uncertain dates, wrote within 100 years of Plato's death). Hypotheses rose and fell and led to more—that is hypotheses not only in the sense of conjectures, but also of axioms and problems and methods and concepts chosen as true, productive, and revealing (*cf*. *Meno* 86–87).

Could Theaetetus and Eudoxus create new theories of irrationals, proportions, and solids, without Plato knowing they conceived and tested and destroyed many hypotheses? Mueller may be quantitatively right that refutation has 'less of a role to play in mathematics than in philosophy' ([1992], p. 187). But if the histories are true then Theaetetus and Eudoxus faced and offered a good many refutations and Plato knew it. See Burnyeat [2000] for developments in mathematics around Plato. But there is frustratingly small evidence from that time.

As a research logician Tait knows hypotheses (including conjectures, axioms, and definitions) change while a subject develops. He depicts dialectic as seeking new deductive foundations for exact sciences ([2002], pp. 24 ff.). This logical project is like dialectic in several ways. It exceeds the science or craft of geometry even while geometers are forced towards it. It has up to our day reached no permanent conclusions. It differs decisively from dialectic, as it does not insist on reaching the good or an unhypothetical knowledge of the whole. And yet it does constantly reach beyond itself. This is going on right now.

The most famous single hypothesis in the history of mathematics is the Euclidean parallel postulate: For any line L and point p not on L, there is exactly one line L' through p not meeting L. By mid-nineteenth century the better geometers all understood there were two consistent 'non-Euclidean' plane geometries. One has every two lines meet. The other has many lines not meeting L through any point p not on L. Call these the *axiomatic* non-Euclidean geometries since they are given by axioms on points and lines rather like Euclid's.

Burnyeat calls the Pythagorean theorem ‘an eternal, context-invariant truth’ ([2000], pp. 41f.). On my view, Plato’s Socrates offers general reasons to doubt this. The theorem rested on unexamined hypotheses and so could change. Whether or not Plato thought it *would* ever change its meaning and its status, it *did*. Geometers before Gauss took it as the plain truth. Today it is merely an equivalent for the parallel postulate. It is true in Euclidean space and not in others, and it only approximates the space we live in. This shift barely began with the axiomatic geometries. The watershed was Riemann’s 1854 dissertation ‘On the hypotheses that lie at the base of geometry’ [1867].

Riemann specifically questioned the hypothesis that a straight line is the shortest distance between two points. Is it true of space around us? Is anything like it true on curved surfaces? Subtle reasoning showed something very like it is true in Riemann’s curved and many-dimensional spaces, called *manifolds*. Riemann suspected Euclid’s hypotheses might yield to some kind of curvature in the case of actual physical space. Logically, these manifolds are entirely different from the axiomatic geometries. They are defined using calculus. Geometrically they include the axiomatic geometries as special cases.

High-school geometry says $x^2 + y^2 = 1$ defines a circle. Algebraic geometry uses more general equations to define more general spaces and has seen a long and contentious series of alternative definitions of ‘space’. See Reid ([1990], pp. 114–117). When Alexander Grothendieck gave his *schemes* in the 1960s as one effort, many experts found them ungeometrical because they violate the hypothesis that a function is determined by its values.[27]

Schemes replaced André Weil’s approach: ‘many people who had devoted a large part of their lives to mastering Weil foundations suffered rejection and humiliation, and to my knowledge only one or two have adapted to the new language’ according to Reid ([1990], p. 115), who works with schemes. There was a sometimes bitter clash over meaning: Which ideas gave meaningful geometry? Which clung to old ways? Which were empty abstraction? In our terms: Would geometers better hypothesize Weil’s varieties, or Grothendieck’s schemes? These are not themselves mathematical questions.

Mathematicians are far from a final approach to geometry or a stably fixed array of approaches. For some of the latest see Cartier [2001]. Can we at least expect a single account of physical space or space-time? Not so far. General Relativitivistic quantum theory is creating radically new geometries. Witten [1998] gives a Fields Medalist view of it.

[27] Notably, some schemes have functions f such that $f(p) = 0$ for every point p of the scheme while yet f is not the 0 function on the scheme, because f has non-zero derivatives in some directions pointing out of the scheme.

Biology suggested René Thom's hypotheses on simplifying geometry [1975]. Many of his methods are now standard. But his hypotheses suffered sharp limitations in theory and in applications. See Aubin [1998]. The collection edited by Engquist and Schmidt [2001] shows mathematics drawing from computer graphics, medicine, finance, crystallography, and really an illimitable range of subjects and practices.

Here is a glimpse of dialectic. And here Plato has cutting force. According to the ideas he gives Socrates (whether they are his own or not) this constant revision of hypotheses is no temporary aberration. It is compulsory, ἀναγκαίος. Science can reach no proper conclusion short of 'the good' or an unhypothetical first principle of everything.

Socrates is not obviously right. Maybe he is wrong. Maybe it is a useless or defeatist way to look at things. It is the opposite of foundationalism. It says mathematics must try to fix its hypotheses within itself, to fix a foundation, but it will compel the soul beyond itself. The finest intellectual achievement Socrates knows (in the dialogues) is the geometer's practice of deduction from hypotheses. But because geometry itself does not also question those hypotheses, it is not truly science (533d). Dialectic is a project not yet an achievement. I find this the opposite of defeatist and useless but I have not tried to prove it. I have tried to prove it is no idle dream.

Apparently in Plato's time, and certainly in ours, the history of geometry was a history of bearing hypotheses like children, destroying them, gathering the dead, appointing them as oracles, and winning them as prizes. This disappears from textbooks for the best of reasons. Proof needs purportedly fixed axioms or definitions. No mathematical theory is dialectical. Yet axioms and definitions do change. This larger process leads to dialectic. And the larger process in geometry has no conclusion in sight. Politeness asks that a philosophic paper have one.

11. Conclusion

The one mathematical platonist in Plato is Glaucon. He and Socrates agree arithmetic and geometry aim at knowing what always is, and compel the soul towards understanding. Socrates says they fail. He never says 'what always is' will include mathematical objects. He says mathematics is necessarily hypothetical even as it compels the soul towards the unhypothetical. He proposes a huge project raising astronomy and harmonics to the level of existing geometry. No one mentions reforming arithmetic or geometry to study eternal objects. Glaucon says they already do that—Plato's brother, one of the 'Sons of Ariston, godlike descendants of a famous man' (368a).

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