Two Constructivist Aspects of Category Theory

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Abstract: Category theory has two unexpected links to constructivism: First, why is topos logic so close to intuitionistic logic? The paper argues that in part the resemblance is superficial, in part it is due to selective attention, and in part topos theory is objectively tied to the motives for later intuitionistic logic little related to Brouwer's own stated motives. Second, why is so much of general category theory somehow constructive? The paper aims to synthesize three hypotheses on why it would be so, with three that suggest it is not.

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The founders of category theory were not sympathetic to constructivism or to the idea that there can be a systematically constructive mathematics. Yet category theory has two unexpected links to constructivism which deserve to be posed as questions rather than stated as facts. Why is topos logic so close to intuitionistic logic? And why is so much general category theory somehow constructive?

The first is formally unproblematic. It is a fact of model theory that in many toposes the axiom of choice is false, the law of excluded middle fails, and the double negation of a sentence does not imply that sentence. The serious question, though, is more conceptual: Why should toposes agree so closely with the logic Brouwer and Heyting created? We weigh several answers which each have some share of the truth: The resemblance is superficial. Or, it is rigged by selective attention. Or, topos theory is objectively tied to the motives for later intuitionistic logic, though not to Brouwer's stated motives. At any rate topos theory has been a productive source of formal models for many aspects of intuitionistic foundations.

Without assuming any precise sense of *constructive mathematics*, we can discuss the relation of category theory to logical themes associated with constructivism: avoiding excluded middle, requiring specified instances for existence claims, avoiding the axiom of choice etc. Obviously there are versions of category theory that meet these restrictions. The question is, why does so much category theory naturally meet them? Very often, once a category theoretic problem is laid out, just one construction from the data is even a candidate solution and in fact it is the solution. We consider six hypotheses:

1. Category theory is so general that nothing weaker than explicit constructions can work across the whole range of it.

2. Category theory conceals non-constructivity in its basic terms.

3. Category theory is too young yet to need nonconstructive proofs.

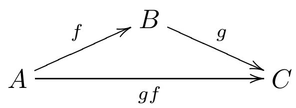
4. Category theory looks constructive because so much of it has been created for computer science.

5. Category theory gives such direct access to structure that it naturally finds explicit solutions to its problems.

6. The category axioms have such a weak logical form that there is little occasion for non-constructive methods.

Hypotheses 1, 5, and 6 offer reasons why category theory would be somehow constructive. Hypotheses 2, 3, and 4 urge that it is not really so. We attempt a synthesis.

The category axioms are few enough to give in full. A category has objects $A, B, C \ldots$, and arrows $f : A \to B$ between them. For example the category of sets, called **Set**, has sets as objects and functions $f : A \to B$ as arrows. In any category **C**, when arrows f and g have matching codomain and domain, as shown, they have a composite gf:



Each object A has an *identity* morphism $1_A : A \to A$ defined as having $f1_A = f$ and $1_B f = f$ for every $f : A \to B$. The last axiom is associativity, saying $(hg)f = h(gf)$ for any $h : C \to D$. Clearly these hold for functions between sets, with 1_A the usual identity function on A, and gf the composite defined by $(gf)(x) = g(f(x))$. Other categories have quite other objects and arrows, see [Mac Lane 1986, chap. XI] or [McLarty 1998].

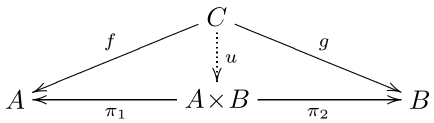
1 Category Theory as somehow Constructive

The 1969 Summer Institute on category theory at Bowdoin College produced a "final exam" with the stern instructions "this is a take-home exam: do not bring it back!" and these two among its questions:

> 4. (Mac Lane's Theorem) Prove that every diagram commutes.
>
> 14. Write down the evident diagram, apply the obvious argument, and obtain the usual result. [Phreilambud 1970]

Question 4 is like saying every equation in calculus is true—except that in calculus it would just be a grim mistake. From introductory calculus through research in analysis one often finds attractive equations which are, sadly, not true. It makes a joke in category theory because of the common experience that every diagram one is tempted to write down *does* commute.[1] Question 14 takes it farther. Many theorems in category theory are proven by explicit, and evident, diagrams.

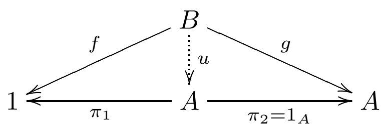
A simple case is typical. A *terminal object* in a category **C** is defined as an object 1 such that for each object A of **C** there is one and only one arrow from A to 1. In the category **Set** the terminal objects are precisely the singleton sets $S = \{s\}$. The *product* of two objects A and B in **C** is defined as an object $A \times B$ plus two *projection arrows*, say $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ with the natural property of projections from a product. That is, for any object C and arrows $f : C \to A$ and $g : C \to B$ in **C**, one and only one arrow $u : C \to A \times B$ makes this diagram commute, or in other words $\pi_1 u = f$ and $\pi_2 u = g$:



Theorem 1 *(In any category* **C** *that has a terminal object 1.) For every object A there are projection arrows $\pi_1 : A \to 1$ and $\pi_2 : A \to A$ so that A is its own product with 1. That is $1 \times A = A$.*

The pattern of the proof recurs throughout category theory:

- We need an arrow $\pi_1 : A \to 1$ with a certain property. But by definition of 1, there is only one arrow $A \to 1$ so it must be π_1.
- We need an arrow $\pi_2 : A \to A$ with a certain property. There may be many different arrows $A \to A$ (as for example a set generally has many functions to itself). But, given no special information about A, only the identity arrow 1_A stands out. So try $\pi_2 = 1_A$.
- For any object B and arrows $f : B \to 1$ and $g : B \to A$ in **C** we need a unique arrow u making the diagram commute:



No arrow $B \to A$ stands out per se, but exactly one is named in the data. That is g, so try $u = g$.

[1] It also refers to Mac Lane's coherence theorems which say that every plausible diagram in a certain situation commutes [Mac Lane 1998, 165].

These choices work. On the left, $\pi_1 u = f$ since there is only one arrow $B \to 1$. On the right, $1_A g = g$ by definition of 1_A. The proof is done.

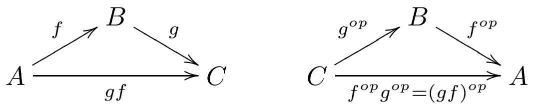
More significant proofs often use the same pattern. We need an arrow with such-and-so domain, and such-and-so codomain, and a certain property. The data offer just one straightforward construction of an arrow with that domain and codomain. And indeed that arrow has the property.[2] These proofs are constructive in the sense that they use no argument by contradiction, nor excluded middle, nor choice. The answer is explicitly constructed from the data of the problem.

1.1 The Six Hypotheses

Hypothesis 1 offers to explain this pervasive fact by a variant of the idea that constructive proof *proves more*. It supposes that only constructive proofs prove *enough*:

Hypothesis 1 Category theory is so general that nothing weaker than explicit constructions can work across the whole range of it.

Theorem 1 looks like it deals with singleton sets and cartesian products. Interpreted in the category **Set** it deals with exactly those. In many familiar categories it deals with things like those. But "like" is a broad term. Take the category $\mathbf{Set^{op}}$, called the *dual* to the category of sets. The objects are sets, and an arrow $f^{op}: A \to B$ in $\mathbf{Set^{op}}$ is a function in the opposite direction $f: B \to A$. The superscript "op" merely shows we take f^{op} as an arrow of $\mathbf{Set^{op}}$, reversed from its direction as a function. Composition is defined in the natural way:



Composing in **Set** Composing in $\mathbf{Set^{op}}$

The empty set $\emptyset$ is terminal in $\mathbf{Set^{op}}$, which is the same as saying it is *initial* in **Set**. Every set A has a unique $\mathbf{Set^{op}}$ arrow $u^{op}: A \to \emptyset$ because it has a unique function $u: \emptyset \to A$, namely the empty function. Similar arrow switching shows that for any sets A, B the disjoint union $A + B$ is a product in $\mathbf{Set^{op}}$. Reverse the inclusion function $i_1: A \to A + B$ to make a projection $i_1^{op}: A + B \to A$ in $\mathbf{Set^{op}}$.

[2]For those who know the terminology: The canonical version uses adjoints. We have an adjunction $\mathbf{F} \dashv \mathbf{G}$ and need an arrow $\mathbf{F}A \to B$ with a certain property. The data offer just one $g: A \to \mathbf{G}B$ and its adjunct has the property.

Theorem 1 applied in $\mathbf{Set^{op}}$ says every set A is its own disjoint union $A + \emptyset$ with the empty set. Other categories give other interpretations. A formal logical theory can be construed as a category with formulas as objects and implications as arrows. In such a category (resp. its dual) the theorem says every formula is equivalent to it conjunction with $True$ (resp. its disjunction with $False$), see e.g. [Johnstone 2002, vol. 2, D1]. More arcane "products," widely used in algebra, number theory, geometry or other fields, appear as categorical products in suitable categories and so Theorem 1 applies to them. Hypothesis 1 suggests that only constructive proofs are strong enough to be valid across all of this range—or at it least it is so to a good approximation.

Hypothesis 2 Category theory conceals non-constructivity in its basic terms.

The argument so far has been that much category theory uses constructive logic: no excluded middle, no double negation or reductio proof. But what about the basic steps, such as composing arrows? Worse, a basic step in many proofs is to take a *limit* or *colimit* over some infinite diagram (e.g. the Freyd adjoint functor theorems [Mac Lane 1998, 127]. Even these proofs are relatively constructive since the theorem explicitly assumes these limits or colimits exist. But there is no constructive proof that they do exist in the desired applications, e.g. to the category of groups or of topological spaces. When so many theorems use such assumptions, can the theory really be called constructive?

Traditional constructivisms have epistemological motives. They take basic constructions of natural numbers as intuitively clear: selecting 0, and passing from a given n to $n+1$. Most constructivists can evaluate any specified primitive recursive function at any given argument—because the evaluation reduces by (what most constructivists regard as) explicit steps to those basic constructions. Theorem 1 was proved by explicit basic steps: given A select its unique arrow to 1, and its identity arrow 1_A; given composable arrows form their composite. But these are not like selecting 0 or passing from n to $n+1$. In axiomatic category theory neither the datum A nor its identity arrow 1_A can be specified at all. And specific examples generally are not specified constructively in any traditional sense.

Even the simplest step, finding the identity arrow of an object A, need not be constructive even when A itself is constructively given and each arrow is effective. For example, assume some effective coding of the partial recursive functions φ_n by natural numbers n. Form a category whose objects are recursive subsets of the natural numbers, and arrows are natural numbers. Specifically, $n : S \to S'$ is an arrow if

- $\varphi_n(x) \in S'$ for every $x \in S$.
- $\varphi_n(x)$ is undefined when $x \notin S$.
- n is the smallest natural number which codes the function φ_n.

Coding any recursive specification of S amounts to coding the identity function restricted to S. But there is no effective way to find the *smallest* code for it. So there is no effective way to find the identity arrow on S. For the same reason, composition is not effective in this category.

In short, much general category theory is relatively constructive. It proceeds explicitly by basic categorical steps. But the steps are not "constructive" in any epistemological sense.

Hypothesis 3 Category theory is too young yet to need nonconstructive proofs.

Hypothesis 4 Category theory looks constructive because so much of it has been created for computer science.

These make a pair since hypothesis 3 was more plausible up to, say, the 1970s, when hypothesis 4 supplanted it. By the 1970s the Grothendieck school had produced huge amounts of category theory applied in topology and algebraic geometry. Others, notably around Mac Lane and Lawvere, had extended the theory in many directions including logic and foundations of mathematics. The subject had grown enough to leave hypothesis 3 behind.

Much of the growth has been in computer science, and especially applications to computable data type specification. See e.g. [Jacobs 2001] [Adámek, Escardó & Hofmann 2003]. Probably, though, this is better seen as following a natural tendency in category theory than as forcing the subject into a computable framework. And of course the theory of computing is not itself all computable.

Hypothesis 5 Category theory gives such direct access to structure that it naturally finds explicit solutions to its problems.

Hypothesis 6 The category axioms have such a weak logical form that there is little occasion for non-constructive methods.

These two make a natural pair if "direct" access means access not mediated by powerful logical tools. Textbooks use category theory as a convenient, light, general framework for organizing many particular kinds of structure [Lang 1965]. Hypothesis 5 suggests this works by foregrounding the explicit, constructive aspects of each particular structure. Hypothesis 6 suggests the key to this is the logical form of the basic category axioms. The various versions of constructive mathematics differ from classical over the use of negation, disjunction, existential quantification, and the axiom of choice. None of these occurs in the category axioms. Only the existential quantifier needs any discussion.

There is no genuine existential quantifier in the axiom saying arrows have domains and codomains, because *every* arrow has a *unique* domain and codomain. The axiom can use a domain operator Dom and a codomain operator Cod so the formula

$$\mathrm{Dom}(f) = A \quad \& \quad \mathrm{Cod}(f) = B$$

says A is the domain of f and B the codomain. Composition too can be expressed by an operator, written as mere juxtaposition gf. The composite is unique when it exists. Composition is a partial operator. The key point is that gf exists if and only if an equation is satisfied:

$$\mathrm{Cod}(f) = \mathrm{Dom}(g)$$

The category axioms form an *essentially algebraic* theory [Freyd 1972].[3] They can be stated by equations using a partially defined operator (composition), with its domain of definition given by equations in preceding operators (the domain and codomain operators). They also use an identity arrow operator 1_ taking an object A to its identity 1_A. It is convenient to present them as sequents with no connectives:

$$\begin{array}{ll} \vdash\ \mathrm{Dom}(gf) = \mathrm{Dom}(f) & \vdash \mathrm{Cod}(gf) = \mathrm{Cod}(g) \\ \vdash\ \mathrm{Dom}(1_A) = A & \vdash \mathrm{Cod}(1_A) = A \\ \vdash\ f1_A = f & \vdash 1_B f = f \\ \vdash\ (hg)f = h(gf) & \end{array}$$

A sequent may have a list of equations on the left hand side, and is read as saying they entail the equation on the right. The axioms need no equations on the left.

The only complication is that a sequent including a composite $\tau_2\tau_1$ implicitly assumes $\mathrm{Cod}(\tau_1) = \mathrm{Dom}(\tau_2)$. So, when a cut eliminates a

[3] Compare [McLarty 1986], [Johnstone 2002, D.1.3].

term $\tau_2\tau_1$ from a sequent, then $\text{Cod}(\tau_1) = \text{Dom}(\tau_2)$ must be added to the left hand side of the resulting sequent. The assumption which had been implicit must be made explicit. It can be cut later if in fact it follows from other explicit assumptions.[4]

Then the only rules of inference are the equality rules, the modified cut rule and term substitution. These are all explicit constructions. The same weak logic suffices for many extension of the category axioms: categories with all finite products, or with all finite colimits, or cartesian closed categories, and much more. See the references. Each of these naturally uses explicit constructions.

This logic is too weak for any foundation for mathematics. Its theories always admit trivial models, so that any candidate foundation must have axioms not in this form. For example the central axiom for categorical set theory says a function is determined by its value on elements. In full: for any two different parallel functions, $f, g\colon A \to B$ with $f \neq g$, there is some $x\colon 1 \to A$ with $f(x) \neq g(x)$. This uses negation. And even so it does not preclude models trivial in the sense that all sets are isomorphic, simultaneously initial (intuitively, empty) and terminal (intuitively, singleton), and for any two sets A, B there is exactly one function $A \to B$. One convenient non-triviality axiom names an initial set $\emptyset$, and a terminal set 1, and says, with the universality interpretation of the free variable f:

$$\vdash \neg[\text{Dom}(f) = 1 \ \& \ \text{Cod}(f) = 0]$$

No arrow goes $1 \to 0$. Then any theorem which requires non-triviality will include some reductio step in its proof, going back to one of these negations in the axioms.

In short, unlike the basic category axioms, the usual categorical foundations cannot be given in a logic of explicit constructions. They can be construed "constructively" of course. But that will make a real difference from the ordinary classical construal.

It remains, though, that much of category theory at every level naturally rests on construction. The hypotheses cooperate in explaining this. To say the axioms are weak is to say they are general. In this sense hypothesis 6 subsumes hypothesis 1 saying category theory is so general that it must be largely constructive. Hypotheses 1 and 5 combine to say it is a natural generality, not just generality in principle. All three happily join hypothesis 2—except that instead of "concealing" stronger logical principles, category theory *locates* them where they belong. They are in the

[4]This is a notational variant of the presentation in [McLarty 1986].

specifics but not in the general framework of each mathematical subject. Hypothesis 3 found category too young to need many nonconstructive proofs, which has been historically implausible for some decades. Better to say that category theory in its organizational role is meant to bring out the simplest aspects of each subject. It is not *too young* but rather *too focussed on explication* to rely heavily on non-constructive proof. From this viewpoint hypothesis 4 loses its explanatory role. Category theory is not explained by its use in computer science. But the viewpoint makes it natural that category theory would be used that way since computer science is all about organization and explicit description.

2 Topos Logic as "Intuitionistic"

A *topos* is any category **E** that shares certain key properties with **Set**, the category of all sets. In a nutshell: A topos **E** has a "singleton" 1, in other words a terminal object. It has a product $A{\times}B$ for any two objects A, B. And each object A has a power object $\mathcal{P}A$, which is the power set of A in the case of $\mathbf{E}{=}\mathbf{Set}$.[5]

So there is a standard interpretation of multi-sorted higher order logic in any topos **E**. Given a multi-sorted higher order language $\mathcal{L}$, interpret each sort σ by an object A_σ. An operator $\phi(x_1, \ldots, x_n)$, with sort σ_i for each argument x_i and sort τ as value, is interpreted as an arrow from the product sort

$$A_{\sigma_1} \times \cdots \times A_{\sigma_n} \longrightarrow A_\tau$$

The power sort $\mathcal{P}\sigma$ of any sort σ is interpreted by the power object $\mathcal{P}A_\sigma$ of the object A_σ. This implies that predicates $P(x)$ on terms x of sort σ correspond to subobjects

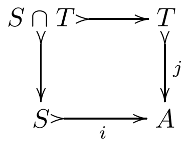
$$S \overset{i_P}{\rightarrowtail} A_\sigma$$

Here a subobject is any monic arrow to A.[6]

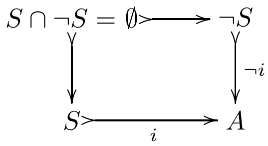
[5]Detailed treatments of topos theory and logic are in [Bell 1988][Johnstone 2002][Mac Lane & Moerdijk 1992][McLarty 1991] among many other sources.

[6]Predicate symbols may be taken as such, or reconstrued as operator symbols with values in a truth-value sort Ω. Subobjects may be construed as monic arrows (treated up to equivalence) or as equivalence classes of monic arrows, or as arrows to a truth value object Ω. No matter which formal definitions are chosen, the interpretants of predicates correspond to subobjects.

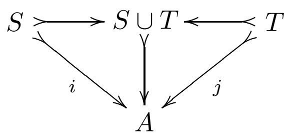
The topos axioms imply there is an empty, or initial, object $\emptyset$. They imply that any two subobjects i, j of an object A have an intersection:



Namely, $S \cap T$ is the largest subobject of A contained in both S and T. Any subobject $i : S \rightarrowtail A$ has a negation, namely the largest subobject of A disjoint from it:



The axioms imply that any two subobjects i, j of an object A have a union defined as the smallest subobject containing both:



These definitions typify topos logic: The intersection of two subobjects has to be smaller than each, and clearly has to be the largest subobject smaller than each. Given that the negation of S should be disjoint from S, surely it must be the largest subobject disjoint from S. The union of two subobjects has to include both, and clearly must be the smallest subobject to include both.

Here is an element of intuitionism in topos logic. These definitions imply only some of the classical laws. The definition says S and $\neg S$ are disjoint. But $\neg\neg S$ is the largest subobject disjoint from $\neg S$, so it is larger than S:

$$S \subseteq \neg\neg S$$

The opposite inclusion does not always hold in topos models, and neither does the law of excluded middle. For example, let $\mathbf{E} = \mathbf{Sh}_{\mathbb{R}}$ be the topos of sheaves on the real line $\mathbb{R}$. The subobjects of 1 (in other words the *truth values* in $\mathbf{E}$) are open subsets of $\mathbb{R}$. Think of a truth value as measuring the *extent* to which a claim is true as it varies over the line. A claim is *True* if its measure is the whole real line and it is *False* if its measure is the empty set. In other words the maximal subset $\mathbb{R} \rightarrowtail \mathbb{R}$ of the line is the truth value *True*, and the minimal subset $\emptyset \rightarrowtail \mathbb{R}$ is the truth value *False*. There are infinitely many intermediate truth values.

Intersection and union of these subobjects in $\mathbf{Sh}_{\mathbb{R}}$ have their meanings naively lifted from **Set** because the set theoretic intersection or union of open subsets is always open. In contrast the negation of an open subset is the largest open subset disjoint from it and this is generally smaller than the set theoretic complement. Take $U = \{x \in \mathbb{R} \mid 0 \neq x\}$. Its set theoretic complement is the singleton $\{0\}$ and not open, so not a subobject of 1. The only open subset disjoint from U is the empty set $\emptyset$. So $\neg U = \emptyset$. But every subset is disjoint from $\emptyset$ so

$$\neg\neg U \;=\; \neg\emptyset \;=\; \mathbb{R} \;\not\subseteq\; U$$

The double negation of U is *True* although U is not. And the union of U with its negation is not *True*:

$$U \cup \neg U \;=\; U \cup \emptyset \;=\; U \;\neq\; \mathbb{R}$$

In terms of logic, sentences of the form

$$\neg\neg\phi \Rightarrow \phi \quad \text{and} \quad \phi \vee \neg\phi$$

are not always true in topos models.

The axiom of choice fails in most toposes. A convenient form of the axiom says: Every onto function $f\colon A \to B$ has a right inverse $g\colon B \to A$. More fully: if for each $y \in B$ there exists an $x \in A$ such that $f(x) = y$ then some function g selects for each $y \in B$ a value such that $f(g(y)) = y$. Classical mathematicians accept this as true for sets. But in the topos $\mathbf{E} = \mathbf{Sh}_T$ of sheaves on a topological space T all functions are continuous and this same axiom says: If a continuous function $f\colon A \to B$ is onto, then it has a continuous right inverse $g\colon B \to A$. Classical mathematicians know this is generally false. The cubic polynomial

$$\mathbb{R} \xrightarrow{x^3 - x} \mathbb{R}$$

is continuous and onto. It has infinitely many right inverses—but none are *continuous*.[7] A classical mathematician pursuing model theory in toposes will find the axiom of choice fails in this case in this topos.

[7] I.e, for every $y_0 \in \mathbb{R}$ there is at least one solution in x to $x^3 - x = y_0$. Indeed there is at least one way to choose solutions to $x^3 - x = y$ continuously for all y in some interval around y_0. For a small interval around $y_0 = 0$ there are three ways. But there is no way to choose them continuously for all $y \in \mathbb{R}$.

So far this is like intuitionistic logic. But a central idea of intuitionism fails in many toposes. An existential statement $\exists x\phi(x)$ can be true in a topos when there is no instance verifying $\phi(x)$. That is, when it is "everywhere true that there is an instance" yet no instance works everywhere. Roughly, in the topological case of a topos $\mathbf{E} = \mathbf{Sh}_T$ of sheaves on a topological space T, the measure of truth of an existential statement $\exists x\phi(x)$ is the union of all the open subsets $U \subseteq T$ such that some instance defined over U makes $\phi(x)$ true. It can happen that T is covered by such subsets yet no one instance exists over all of T. Then $\exists x\phi(x)$ is true over all of T but no instance of it is.

Further, as in the case of general category theory, topos logic does not require its predicates and operations to be any kind of intuitionistic. Arithmetic in a topos uses induction on every formula $\psi(x)$ with x a variable over the natural numbers. Given x a variable over an object A, every formula $\phi(x)$ with x as the sole free variable defines a subobject of A. Topos logic includes none of the intuitionistic restrictions on induction or comprehension found, for example, in [Troelstra 1973].

The subject *synthetic differential geometry* (SDG) posits an ordered continuum R like the real number line $\mathbb{R}$ but with this property quite unlike $\mathbb{R}$: For any function $f: R \to R$ and any point $x_0 \in R$ there is a unique linear function $ax + b$ which agrees with f on an infinitesimal neighborhood of x_0. In other words, every function $f: R \to R$ has a uniquely defined derivative at each point. This theory is classically inconsistent but it has topos models, see [Bell 1998]. In these models a function $f: R \to R$ need not be intuitionistically specifiable, nor is there any constructive or intuitionistic procedure for finding the derivative of f at a given point. In these models of SDG the classical laws of excluded middle and double negation fail. So does the axiom of choice. But the axioms of the theory make no intuitionistic sense.[8]

Some toposes are closer than others to classical intuitionism. In some an existential statement $\exists x\phi(x)$ is true only when some instance of it is. In other toposes this principle fails in general but holds for some interesting class of predicates $\phi(x)$. Topos models are useful in exploring the foundations of Brouwer's intuitionistic analysis, notably continuity principles and choice sequences [Fourman 1979][van der Hoeven, Gerrit & Moerdijk 1984]. Topos logic per se is far from Brouwerian.

The resemblance of topos logic to intuitionism is superficial from Brouwer's viewpoint. Brouwer would construct mathematics from "the basic intuition of two-ity" [Brouwer 1911, 122]. Topos logic has none

[8]It is a different point, though also correct, that the models are not intuitionistic.

of that. From the viewpoint of formal model theory, exploring aspects of Brouwer's mathematics without accepting his epistemology, there is similarity in just some respects. Yet the question remains: Why is topos logic even this close to Heyting's formalization of Brouwer's ideas, while sharing so little of the stated motivation?

The examples above suggest a natural idea. Brouwer was a topologist and many toposes are "topological.", i.e. toposes $\mathbf{E} = \mathbf{Sh}_T$ of sheaves on any topological space T. Excluded middle, and double negation, and the axiom of choice fail in such toposes. They fail basically because they are not stable under continuous variation. A "truth value" in such a topos is an open set $U \subseteq T$, so any point p in U is surrounded by some little region lying entirely in U. Perhaps Brouwer favored claims that are stable in this way? For example, perhaps he did not want to affirm "every real number is either equal to 0 or not" because 0 has non-zero reals arbitrarily close to it? This explanation is doubly inadequate, though, as there is little trace of it in Brouwer's own topology and topology is not the only source of topos logic.[9]

Brouwer in no way avoids classical logic in his topology and even gratuitously uses contradiction in the proof central to his method.[10] He produces four integers we can call a_1, a_2, a_3, a_4 and he needs to prove $a_1 = a_4$. He first proves

$$a_1 = a_2 \quad \text{and} \quad a_2 = a_3 \quad \text{and} \quad a_3 = a_4$$

He then supposes $a_1 \neq a_4$, reasons from this back to $a_1 \neq a_2$, and from the contradiction concludes $a_1 = a_4$ [Brouwer 1911, 432]. He frequently assumes every point in $\mathbb{R}^n$ lies either on or not on any given $(n-1)$-dimensional plane. As a special case: every real number is either 0 or not 0. His later intuitionism would actually support argument by contradiction for equality of integers, and reject it for the reals. But in 1911 he used both with no hint at any such distinction.

His great topological papers follow the line of his dissertation: Excluded middle and proof by contradiction are as harmless as modus ponens, but attention to their use is harmful:

> the proposition "A function is either differentiable or not differentiable" says *nothing*; it expresses the same as the follo-

[9]Indeed the original source was *Grothendieck topologies* in algebraic geometry. These are not topological spaces in the usual sense at all [Mac Lane & Moerdijk 1992].

[10]I.e. a lemma using homotopy to show mapping degree is constant on non-singular points of certain maps $\mathbb{R}^n \to \mathbb{R}^n$.

> wing: "If a function is not differentiable then it is not differentiable."
>
> But the logician, looking at the *words* of the former sentence, and discovering a regularity in the combination of words in this and similar sentences, here again projects a mathematical system, and he calls such a sentence an application of the *tertium non datur* [excluded middle]. . . .
>
> It is self-evident that in the language that accompanies mathematics, the succession of words obeys certain laws, but to consider these laws as directing the building up of mathematics, it is there that the error lies.[11]

So Brouwer's specific work in topology does not seem closely tied in his own mind to his later critique of logic.

Furthermore the resemblance between intuitionist and topos logic is not restricted to the topological case. It is common to all toposes. That means all elementary toposes, all models of the Lawvere-Tierney topos axioms, and not only the Grothendieck toposes. Heyting's intuitionistic predicate logic is sound and complete for models in all elementary toposes [Bell 1988]. Of course it is incomplete when the models are confined to certain toposes, notably when they are required to be in the topos **Set** or in other words are required to be standard Tarski type models.

Actually closer to Brouwer's intuitionistic analysis are *realizability toposes*.[12] Introduced around 1980, they are based on an idea already linked to Brouwer by [Kleene 1945]. In the usual realizability toposes an existential sentence $\exists x \phi(x)$ is true if and only if some instance of it is. But for a universal, say $\forall x \phi(x)$, it is not enough that each specific value x_0 have $\phi(x_0)$ true. There needs to be some suitably computable procedure taking each value x_0 and finding a *realizer* for $\phi(x_0)$. The motivation is that a realizer for a sentence codes a proof of that sentence. In particular a quantified disjunction

$$\forall x[\phi(x) \vee \neg\phi(x)]$$

is true in such a topos only if there is a suitably computable procedure taking each value x_0 and either finding that $\phi(x_0)$ is true or else finding that $\neg\phi(x_0)$ is true. There must be a decision routine.

[11] [Brouwer 1907, 75, 90], Brouwer's emphasis. Compare contradiction and the syllogism on pages 73–74.

[12] These are not Grothendieck. See [Hyland 1982][McLarty 2001][van Oosten 2004].

To explain why topos logic is this close to intuitionistic logic we must begin by distinguishing formalized intuitionistic logic from Brouwer's philosophy. The more familiar issue is the axiom of choice. The most telling for us will be in Heyting's intuitionistic sentential logic.

Brouwer did not criticize the axiom of choice so much as the classical applications of it. For an intuitionist, given a function $f : A \to B$, the only way to know that for each $y \in B$ there exists an $x \in A$ with $f(x) = y$, is to know some construction which takes an arbitrary $y \in B$ to such an x. I.e. the only way to know f is onto is to know a right inverse g to it. The axiom of choice from this point of view is a pure tautology, of no use, since the consequent merely repeats the antecedent. Yet formal systems of "intuitionistic set theory" generally reject it. The issue is well laid out in [DeVidi 2004].

As to sentential logic, Heyting's intuitionistic version includes as true

$$\phi \Rightarrow (\neg\phi \Rightarrow \psi)$$

This is nonsense in Brouwer's philosophy because it reasons from explicitly absurd premises: if you know ϕ then if you also know $\neg\phi$ then.... Kolmogorov excluded it from his formalization of Brouwerian logic for just this reason [Kolmogorov 1967]. Later Heyting included it with some discomfort. This issue is described in detail in [Haack 1996, 102].

In each case topos logic agrees with the usual formalized versions against Brouwer's philosophical tendency—for example with Heyting in sentential logic rather than Kolmogorov, though Kolmogorov is closer to Brouwer. The best explanation is Lawvere's view of a topos as a universe of *variable sets* [Lawvere 1975][Lawvere 1976]. Lawvere points out that Heyting's intuitionistic predicate logic has a clear explanation as the objective logic of the simplest kind of variation, namely Kripke models where sets vary over a partially ordered set of *stages*. For example, for a negation $\neg\phi$ to be true at a stage p it is not enough that ϕ is not true at p—it requires that ϕ is not true at any stage q later than p.[13] Each model in Kripke's sense gives a topos model. This has a clear analogy to Brouwer's idea of knowledge developing over time—but Brouwer's idea raises ambiguities as above. The objective idea of *presheaf on a partial order* raises none. The influential versions of formal intuitionistic logic have repeatedly followed the objective idea.

The truth conditions of statements in a topos vary progressively along the stages of a Kripke model, or continuously over a topological space,

[13] Kolmogorov's logic has a more complicated semantics where $\neg\phi$ only requires that ϕ is not true at certain later stages [Segerberg 1968].

or computably over a domain of realizers, or in other ways. Excluded middle $\phi \vee \neg\phi$ fails in a given topos when the variation from ϕ to $\neg\phi$ is not neatly split over stages, or separated over the space, or computable by the realizers, or otherwise suitable in the other cases. The axiom of choice fails when the instances to be chosen do exist but do not vary progressively or continuously or computably.... Other classical principles fail in general for analogous reasons. Of course they do all hold in some cases, notably the topos **Set**. As Lawvere puts it, **Set** is the limiting case of constancy, or in other words null variation, so that it poses no obstacle to excluded middle or choice or other classical principles.

To be precise, Heyting's first order predicate logic is sound for all toposes: When premises entail a conclusion in this logic and the premises are true in some topos model $\mathcal{M}$, then the conclusion is true in $\mathcal{M}$. This logic is also complete for topos models: Any set of sentences Γ which entails no contradiction in this logic has a topos model. Indeed the logic is already complete for Kripke models.

We have only looked at a few special kinds of topos. The far more general kinds of variation found in toposes do not extend the range of first order models. They radically extend the range of higher order models, including models of analysis. But there is no standard higher order intuitionistic logic. See the many variants in [Troelstra 1973]. There is no point in asking for a precise comparison of topos logic with "intuitionistic logic" at this level.

Brouwer in 1907 found it harmful to focus on logic. By 1908 he accused science of "the fundamental sin of apprehension or desire" compounded by an "irreligious separation" of means and ends, so that "like any irreligious consciousness, science has neither religious reliability nor reliability in itself... [and so] logical deductions are unreliable in science" [Brouwer 1908, 107]. These motives could not lead to any specific logic in place of classical. Yet as Brouwer shifted from criticizing formal mathematics into producing an intuitionist alternative he thought more and more of logic. He encouraged Heyting to formalize an intuitionistic logic despite Brouwer's well known misgivings [Stigt 1990, 285–92]. This logic needed some more positive basis than religious and epistemological distrust. Heyting focussed on first order logic, where he in fact captured a general sense of stability under variation. He was certainly aware of stability under topological variation, and under computable variation (later formalized in Kleene's realizability), and under variation along posets of "stages of knowledge" (later formalized in Kripke models). Each one of those is in fact general enough to produce the first order logic of toposes in general.

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