Stochastic Synchronization of Type I vs Type II Neural Oscillators: Fokker-Planck studies with the Finite Element Method

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Abstract
We have investigated the effect of the phase-response curve on the dynamics of oscillators driven by noise in two limit cases that are especially relevant for neuroscience. Using the finite element method (FEM) to solve the Fokker-Planck equation we have studied: 1) the impact of noise on the regularity of the oscillations quantified as the coefficient of variation; 2) stochastic synchronization of two uncoupled phase oscillators driven by correlated noise and 3) the ability to predict the phase of one oscillator by knowing the phase of the other quantified as their cross-correlation function. We show that in general, the limit of type II oscillators is more robust to noise and more efficient at synchronizing by correlated noise than type I.

Figure 1 (a) Phase response curves for two limit cases. Left, type I excitability or neural integrator; right, type II excitability or neural resonator. (b) Phase evolution of two oscillators (black and red lines) and threshold crossings (i.e. spikes as black and red dots) for different levels of input correlation, r. The number of synchronous spikes clearly increases with r and appears to be larger for type II excitability than for type I (right) than for type I (left), as confirmed in Figure 2. Equation parameters: ω0=ω1=1, σ=σ1=σ2=1.

Figure 2: Stationary probability density, P(xt,yt), on a color scale for different values of the input correlation, r. The probability of the synchronous states, ϕt=ϕ1-ϕ2 increases as r increases and is larger for type II than for type I for each r. Parameters: ω0=ω1=1, σ=σ1=σ2=1.

Quantification of synchrony as the correlation coefficient of the phases, ρ = (E(ϕ1ϕ2)-E(ϕ1)E(ϕ2))/σ1σ2.

Figure 3: (a) Probability density of the phase difference, P(Δϕ) (thick lines). The probability of being around Δϕ=0 is larger for type II than for type I for any intermediate value of r. This result is in agreement with the results obtained from numerical integration of the stochastic differential equation, shown as thin lines with markers. (b) Cross-correlation function, C(t), normalized in such a way that C(0)=1 gives the correlation coefficient, ρ.

Coefficient of variation, CV
We define the CV as the standard deviation of the cycle duration divided by the mean cycle duration. Thus, the lower the CV, the more robust the oscillations. The CV can be efficiently calculated with the FEM from the temporal moments of the backward (adjoint) Fokker-Planck equation for a single oscillator, as shown below. Figure 4a displays the CV as a function of the amplitude of the input noise, c. The CV increases roughly linearly with c for both type I and type II. However, it is clearly lower in the later case, indicating that the oscillations of type II are more robust to noise than those of type I oscillators.

Fokker-Planck operators and equations
In our analyses we use the forward Fokker-Planck, Lφ, and its adjoint, the backward Fokker-Planck operator, Lφ∗.

Lφφ = −ω0∂2φ + σ2φ2(1 - φ2) + ϕ∂2φφ, Lφφ = −ω0∂2φ - σ2φ2(1 - φ2) + φ∂2φφ.

When using the FEM, these operators are represented as matrices facilitating the numerical analyses. To calculate the stationary probability density, P(xt,yt), we solve the equation LφP(xt,yt)=0 with periodic boundary conditions on ϕ and ϕt. The results are plotted in Figure 2. Then, from P(xt,yt) it is easy to calculate the probability density P(Δϕ) of the phase difference Δϕ=ϕ1-ϕ2, shown in Figure 3a.

Stochastic dynamics
Experimental and computational studies show that neurons receiving correlated stochastic inputs fire synchronous spikes [1]. The stochastic dynamics of two uncoupled neural oscillators driven by correlated noise (Fig.1b) is given by:

where η is the instantaneous phase of the oscillating ith neuron, Ω is the average angular firing frequency, Z(ϕ) is the phase-response curve of the neuronal oscillator (Fig.1a), which can be estimated experimentally [3], describing the change in phase as a function of the phase at which an input arrives. The η(t) are zero-mean, white-noise stochastic inputs, c0(t)=c0=0=0, that are spatially correlated c0(t0)=c0c0/c0=0, thus the correlation coefficient of the inputs is r=c0c0/c0c0=0.

Figure 4: (a) Coefficient of variation of the cycle duration as a function of the noise amplitude, C(t) (circles, SDE). (b) Stochastic synchronization quantified as the correlation coefficient of the phases, r increases with increasing input correlation, r but it is systematically larger for type II than for type I (dashed lines, FEM; solid lines, SDE). (c) Effect of the difference of the intrinsic frequencies on stochastic synchronization. As the difference increases, synchrony deteriorates quickly but more rapidly for type I. (d) The noise amplitude, σ, does not affect synchrony as long as the phase oscillator approximation is valid (σ=1, larger than the average of the term with ω).

Cross-correlation function
The FEM also allows us to calculate the cross-correlation function, C(t) (Fig.3b). The cross-correlation function quantifies the ability to predict the phase of one oscillator by knowing the phase of the other. The cross-correlation function of the phase difference, Δϕ=ϕ1-ϕ2, is given by:

with normalization condition:

CV = [00]<p(xt,yt)> / ∑ λ > 0 λ <p(xt,yt)> = 1.

References

http://www.andrew.cmu.edu/user/fgalan/home.htm