Foci of Algebraic Curves

November 7, 2008

Outline of Talk

- Historical overview, Definition of foci
- Example: k-ellipses
- Example: lemniscates
- Example: critical points of polynomials
- Example: critical points of non-Euclidean polynomials
- Example: numerical range and eigenvalues
- Example: Schwarz reflection

What are Foci?

Apollonius of Perga introduced foci of ellipse and hyperbola (3rd century B.C.E.)
Pappus found the focus of a parabola (4th century C.E.)
Kepler named foci and developed their properties (17th century)
Plücker (1832) defined foci of higher order curves

Plücker Formulation

\( \gamma \) is an algebraic curve, given by \( P(x, y) = 0 \)
where \( P \) is a polynomial of degree \( n \) (and assume real coefficients)

Example: The circle \( (x-a)^2 + (y-b)^2 - r^2 = 0 \)

We think of \( \mathbb{R}^2 \subset \mathbb{C}^2 \subset \mathbb{C}P^2 \)

Replace \( P(x, y) \) with homogeneous polynomial \( P(x, y, z) = 0 \) and extend to a complex curve \( \Gamma \)
in \( \mathbb{C}P^2 \)

Example: \( (x - az)^2 + (y - bz)^2 - r^2 z^2 = 0 \)

Note that every circle passes through the "circular points" \([1, i, 0]\) and \([1, -i, 0]\).

A line through a circular point tangent to \( \Gamma \) is an "isotropic tangent". With equation \( x + iy = (a + ib)z \) (or \( x - iy = (a - ib)z \))
The line meets the "real" plane at the focus \( (a, b) \)
In the case of the circle, this actually gives the center of the circle.
The example of the circle is special, because the isotropic tangent is actually tangent \textit{at the circular point}. This kind of focus is special and may be called a \textit{singular focus}. A real curve which passes through the circular points is a "circular curve".

A quadratic curve which is not a circle is not circular, so it has ordinary foci.

A curve $\Gamma$ has \textit{class} $m$ if there are $m$ tangent lines from an arbitrary point $P$ to it. Such a curve will have at most $m$ foci; a circular curve will have fewer.

**Example 1: The $k$-ellipse**

Given points $P_1, P_2, \ldots, P_k$ in the plane and a positive number $r$, the set of points the sum of whose distances from the $P_i$ is $r$ is called a \textit{$k$-ellipse} or \textit{poly-ellipse}. They were first studied by Tschirnhaus in 1695.

The $k$-ellipse is given by a polynomial of degree $2^k$ if $k$ is odd or degree $2^k - \left( \frac{k}{2} \right)$ if $k$ is even.

The points $P_i$ are foci (though in general there are other foci).
Example 2: Lemniscates

Given points $P_1, P_2, \ldots, P_k$ in the plane and a positive number $r$, the set of points the product of whose distances from the $P_i$ is $r$ is called a (Polynomial) lemniscate or, when $k = 2$, a Cassini oval.

The lemniscate is given by a polynomial of degree $2k$: If

$$P(z) = \prod_{i=1}^{k} (z - P_i) = u(x, y) + iv(x, y)$$

then the equation is $u^2 + v^2 = r^2$.

The points $P_i$ are foci (though in general there are other foci).
Example 3: Siebeck's Theorem

**Theorem:** The zeros of the function

\[ F(Z) = \sum_{i=1}^{p} \frac{m_i}{Z - Z_i} \]

are the foci of the curve of class \( p - 1 \) which touches each line segment \((Z_i, Z_j)\) in a point dividing the line segment in the ratio \( m_i : m_j \).

In particular, the critical points of a polynomial \( P(Z) \) of degree \( p \) are the foci of a curve of class \( p - 1 \) which is tangent to the lines joining pairs of roots of \( P \).

Bôcher-Grace Theorem

If \( P(z) = z^3 + a_2 z^2 + a_1 z + a_0 \) has roots \( R_1, R_2, \) and \( R_3 \), then its derivative \( P'(z) = 3z^2 + 2a_2 z + a_1 \) has its roots at the foci of an ellipse tangent at the midpoints to the three sides of the triangle with vertices at \( R_1, R_2, \) and \( R_3 \).

Class Three

A curve of class three has the property that there are three tangent lines from any point to the curve. Such a curve must look something like this:

![Class Three Curve](image)

The corners of the quadrilateral are the poles of

\[ F(Z) = \sum_{i=1}^{4} \frac{m_i}{Z - Z_i} \]

and the foci are the zeroes.

Application: View

\[ \sum_{i=1}^{p} \frac{m_i}{Z - Z_i} = \sum_{i=1}^{p} \frac{m_i}{|Z - Z_i|^2} (Z - Z_i) \]

as velocity vector field of an incompressible fluid flow. Then \( Z_i \) is a source of strength \( m_i \) (or sink if \( m_i < 0 \)), and the foci are the stagnation points of the flow.

Siebeck's theorem says that given four points the resulting curve of class three will be tangent to the line joining any two of the points. Here is a sample picture of this phenomenon:

![Siebeck's Theorem](image)
Example 4: Non-Euclidean polynomials

A non-Euclidean polynomial (Walsh, 1952), is a function of the form

$$B(Z) = \lambda \prod_{k=1}^{n} \frac{Z - A_k}{1 - A_k Z}, \quad |\lambda| = 1, \quad |A_k| < 1$$

This is an $n$-to-one map from the closed unit disc $D$ to itself (a finite Blaschke product) with $n$ zeros in the interior $D$, and has modulus unity on $C: |Z| = 1$.

Theorem. Let $B(Z) = \lambda \prod_{k=1}^{n} \frac{Z - A_k}{1 - A_k Z}, \quad |\lambda| = 1, |A_k| < 1, n > 2$ be a non-Euclidean polynomial, and let $\gamma$ be the curve in $D$ which is the envelope of the non-Euclidean geodesics (with respect to the Poincaré metric) joining pairs of points $W_i, W_j$ on $C$ satisfying $B(W_i) = B(W_j)$. Then $\gamma$ is (part of) an algebraic curve whose real foci are the critical points of $B(Z)$ in $D$ together with their inverses with respect to $D$.

In the case $n = 3$ the curve is a non-Euclidean ellipse (together with its reflected image). The sum of the hyperbolic distances from the two foci to a point on the curve is constant. It is inscribed in ideal triangles whose vertices $W_1$ satisfy $B(W_1) = B(W_2) = B(W_3)$.

The equation for this curve, after a Möbius transformation, can be given as

$$\Gamma : \frac{4x^2}{a^2} + \frac{4y^2}{b^2} - (x^2 + y^2 + 1)^2 = 0$$
**Example 5: Eigenvalues**

The field of values $W(A)$ of an $n \times n$ matrix $A$ is defined by

$$W(A) = \{ x^* A x : \|x\| = 1 \}$$

$$= \{ \frac{x^* A x}{x^* x} : \|x\| \neq 0 \}$$

It is a compact convex subset of the plane containing the eigenvalues of $A$. (Toeplitz - Hausdorff)

**Kippenhann’s Theorem**

**Theorem.** The boundary $\Gamma$ of the numerical range of an $n \times n$ matrix $A$ is the convex hull of the curve whose equation in line coordinates $(u, v, w)$ is given by

$$\Phi(u, v, w) = \text{det}(uH + vK + wI) = 0$$

where $A = H + iK$ and $H$ and $K$ are Hermitian matrices. Thus, $\Gamma$ is the dual curve of the curve $\Phi(u, v, w) = 0$

**Corollary.** The eigenvalues of $A$ are the foci of the curve $\Gamma$.

**Proof.** The line $ax + by + cz = 0$ is tangent to $\Gamma$ if and only if $\phi(a, b, c) = 0$. The isotropic line $x + iy = (z_0)\bar{z}$ is tangent to $\Gamma$ iff

$$\text{det}(H + iK - z_0 I) = \text{det}(A - z_0 I) = 0$$

Eigenvalues (red) and $\Gamma$ (solid) for $100 \times 100$ grccr matrix

**Note:** The dual curve $\phi(x, y, z) = 0$ is given by a determinant; it is an "RZ curve", important in applications to nonlinear optimization. The solution by Helton and Vinnikov of the Lax Conjecture is the converse: the dual of an RZ curve of degree $n$ is the boundary curve for the numerical range of an $n \times n$ matrix.

An RZ curve of order $n$ has the property that every line through the origin meets the curve in $n$ real points.

**Example of an RZ curve**

$$(x^2 + 8x + y^2 - 12y + 11)(x^2 - 17) = 0$$

**The dual curve**

$$11025x^6 + x^5 + 53x^4 + 79x^3 + 11x^2 - 297x - 1729 = 0$$

$$+9864x^4 + 3912x^3 + 112x^2 + 18x^2 + 256= 0$$
The dual of this RZ curve is not convex, but its convex hull is a numerical range

Example 6: Schwarz Reflection

Let \( \Gamma \) be a curve in \( \mathbb{R}^2 \) given by the algebraic equation \( \phi(x, y) = 0 \). (Edouard) Study defined reflection of points across \( \Gamma \) as follows:

If \( P = (x_1, y_1) \) is any point in the plane, we identify it with the point in \( \mathbb{CP}^2 \) with homogeneous coordinates \( [x_1, y_1, 1] \). There are two lines, \( R_1 \) and \( B_1 \) given in homogeneous coordinates \( [X, Y, Z] \) by

\[
R_1 : X + iY = (x_1 + iy_1)Z \\
B_1 : X - iY = (x_1 - iy_1)Z
\]

Let \( (X_2, Y_2) \) be a point in \( \mathbb{C}^2 \) lying on the line \( R_1 \) and satisfying the equation \( \phi(X_2, Y_2) = 0 \). There will be \( n \) choices of such a point if \( \phi \) has degree \( n \).

(This reveals the fact that Schwarz reflection is only locally defined in a 1–1 manner.)

Now take the line

\[
B_2 : X - iY = (X_2 - iY_2)Z
\]

Let \( Q = (x_3, y_3) \) be the point of intersection of \( B_2 \) with the real plane. \( Q = R(P) \) is the Schwarz reflection of \( P \).
The Schwarz function of \( \Gamma \) is the analytic function such that the curve is the analytic function locally defined (near a given point \((x_0, y_0) \in \Gamma\)) by \( \bar{z} = S(z) \).

\[ S(z) = z \text{ for the real line } y = 0 \]
\[ S(z) = \frac{r^2}{z} \text{ for the circle } x^2 + y^2 = r^2 \]

Schwarzian reflection in \( \Gamma \) is locally characterized as the unique antiholomorphic map \( \mathcal{R} \) fixing points of \( \Gamma \).

It is given by the formula \( \mathcal{R}(z) = \frac{S(z)}{S(\bar{z})} \) for \( z \) near \( x_0 = x_0 + iy_0 \).

The branch points of \( S(z) \) are the foci of \( \gamma \).

Application: Quadrature Domains

\( \Omega \) is a quadrature domain if there are finitely many points \( a_1, a_2, \ldots, a_m \in \Omega \) and coefficients \( c_{ijk} \) such that for any integrable analytic function \( f \)

\[ \int_{\Omega} f \, d\lambda = \sum_{k=1}^{m} \sum_{j=0}^{m-1} c_{ijk} f^{(j)}(a_k) \]

Why? Inversion carries the ordinary foci of \( \gamma \) to the foci of \( \gamma^{-1} \) (and vice-versa). Since \( \gamma \) contains all of its foci, its inverse contains no foci! So \( S(z) \) has no branch points inside \( \gamma^{-1} \).

Example: The ellipse is not a quadrature domain. But the inverted ellipse is a quadrature domain: the Neumann quadrature domain, the first historical example of such a region.

Conclusion: Foci are interesting

Theorem: [Davis, Aharonov - Shapiro] A bounded domain \( \Omega \) is a quadrature domain if and only if the Schwartz function is meromorphic on all of \( \Omega \).

Theorem: Let \( \mathcal{N} \subset \mathbb{C} \) be the numerical range of an \( n \times n \) complex matrix \( A \). Assume all eigenvalues of \( A \) lie in the interior of \( \mathcal{N} \), and the boundary of \( \mathcal{N} \) is a nonsingular, algebraic curve \( \gamma \). Assume also that 0 is \( \in \mathcal{N} \). Then \( \gamma^{-1} \), obtained by inversion in the plane, bounds a quadrature domain \( \mathcal{D} \).
The End