Solutions

# Math 224 Exam 1
September 12, 2012

15 1. (a) Solve the initial value problem

$$\frac{dy}{dt} = y^2 t, \qquad y(0) = 1.$$

Using separation of variables:

$$\int \frac{dy}{y^2} = \int t\,dt \Rightarrow \frac{-1}{y} = \frac{t^2}{2} + c$$

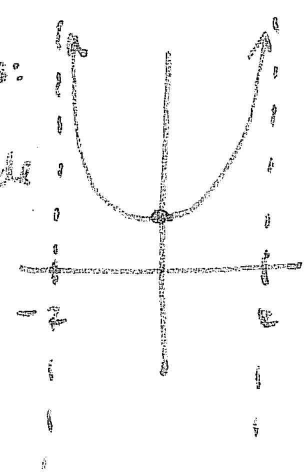
$$\Rightarrow y = \frac{-2}{t^2 + C}$$

$y(0) = \frac{-2}{0+C} = -\frac{2}{C}$. So $y(0) = 1 \Rightarrow C = -2$

$$\Rightarrow y(t) = \frac{2}{2-t^2}$$

10 (b) What is the domain of definition of your solution? What happens as $t$ approaches the limits of that domain?

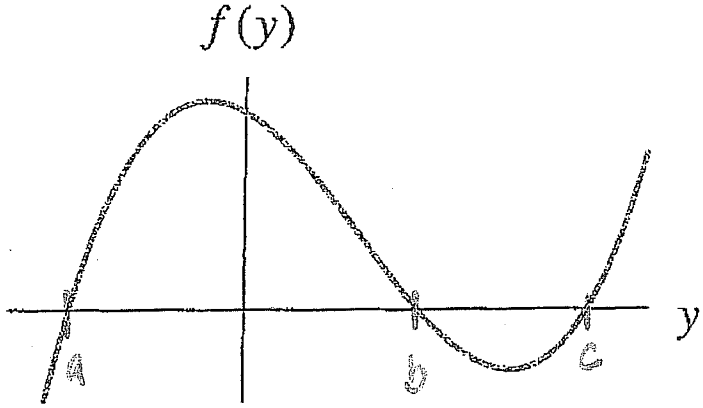
The domain of definition of solution to the I.V.P. above is $t \in (-\sqrt{2}, \sqrt{2})$. The solution there looks like this:




As $t \to \sqrt{2}$ or $t \to -\sqrt{2}$, (from inside $(-\sqrt{2}, \sqrt{2})$)
$y \to \infty$.

Note: $(2, \infty)$ and $(-\infty, -2)$ are not in the domain of

25 2. Suppose the following is a graph of the function $f(y)$.




Sketch the slope field of the differential equation

$$\frac{dy}{dt} = f(y),$$

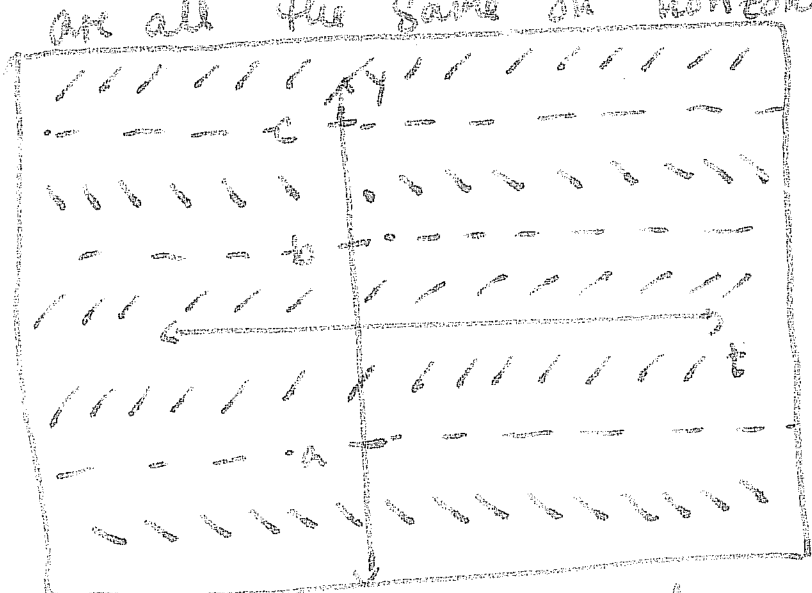
and describe the possible long term behaviors of the solutions, depending on their initial conditions.

(**Suggstion**: label important points on the axes of the graph above.)

Long-term behaviors:

From the slope field below, we see that If $y_0 > c$, then $y(t) \to \infty$ as $t \to \infty$ (we don't know if this happens in finite time or not). If $b < y_0 < c$, then $y(t) \to b^+$ as $t \to \infty$. If $a < y_0 < b$, then $y(t) \to b^-$ as $t \to \infty$, and if $y_0 < a$, $y(t) \to -\infty$ as $t \to \infty$. If $y_0 \in \{a, b, c\}$, then $y$ is constant.

Note that the differential equation above is autonomous, so that slope lines in the slope field are all the same on horizontal axes.




For $y \in \{a, b, c\}$, the graph above says that the slope lines are horizontal (slope=0). The slope of solutions is positive in $(a, b)$ and $(c, \infty)$ and negative in $(-\infty, a)$ and

* Note: the height of the curve also

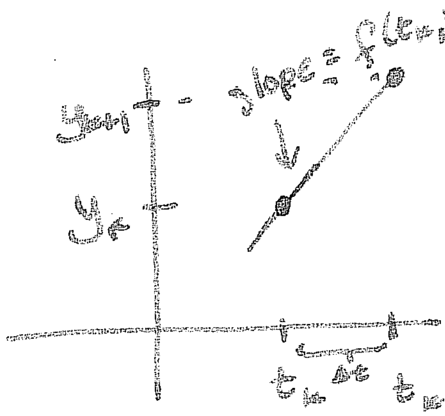
3. (a) State the Euler's method formula for $y_{k+1}$ in terms of $t_k$, $y_k$ and $\Delta t$ when approximating the solution to the initial value problem

15

$$\frac{dy}{dt} = f(t, y) \quad y(t_0) = y_0.$$

Sketch a graph that demonstrates where this formula comes from.

Euler's formula: $y_{k+1} = y_k + (\Delta t) f(t_k, y_k)$




The equation for the line through $(t_k, y_k)$ with slope $f(t_k, y_k)$ is $y = y_k + (t - t_k) f(t_k, y_k)$

We get $y_{k+1}$ by taking the point on this line with t-coordinate $t_{k+1} = t_k + \Delta t$.

(b) Use Euler's method with $\Delta t = 1$ to approximate the solution of

10

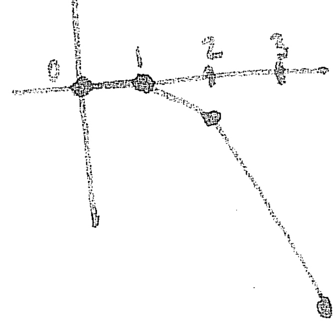
$$\frac{dy}{dt} = 2y - t, \qquad y(0) = 0$$

Note: $f(t, y) = 2y - t$

up to $t = 3$.

| $k$ | $t_k$ | $y_k$ | $f(t_k, y_k)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | -1 |
| 2 | 2 | -1 | -4 |
| 3 | 3 | -5 | |

Approximate sketch of $y$:




$y(3) \approx -5$

4. The behavior of the population of deer in a particular wooded area is modeled by the logistic equation

$$\frac{dP}{dt} = \frac{1}{10} P \left(1 - \frac{P}{2}\right),$$

where $P$ is the population in thousands, time is measured in days.

(a) Suppose that a fraction $\alpha$ of the population is hunted each day; modify the differential equation to reflect this.

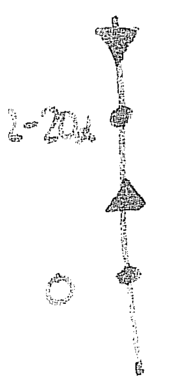
5

This adds a term $-\alpha P$ to $\frac{dP}{dt}$:

$$\frac{dP}{dt} = \frac{1}{10} P\left(1 - \frac{P}{2}\right) - \alpha P$$
$$= \frac{1}{10} P\left(1 - 10\alpha - \frac{P}{2}\right).$$

(b) Suppose first that $0 < \alpha < \frac{1}{10}$.

i. Find all equilibria and sketch the phase line. Include only the relevant range $P > 0$.




5

$P$ is an equilibrium iff $P(1 - 10\alpha - \frac{P}{2}) = 0$. This happens iff $P = 0$ or $P = 2 - 20\alpha$ (which is possible since $1 - 10\alpha > 0$). If $P \in (0, 2-20\alpha)$, then $\frac{dP}{dt} > 0$ and if $P > 2 - 20\alpha$, then $\frac{dP}{dt} < 0$.

ii. Use qualitative analysis to predict the fate of the deer population if the population is initially 5 thousand deer (i.e., $P(0) = 5$).
(**Note**: Even though the equilibrium from the previous part is in terms of $\alpha$, you can still tell how it compares to 5.)

3

Observe:

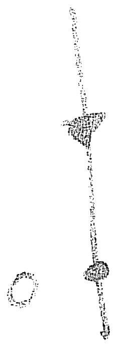
$$2 - 20\alpha > 2 - 20\left(\tfrac{1}{10}\right) = 0 \quad \text{and}$$
$$2 - 20\alpha < 2 - 20(0) = 2.$$

So $P(0) = 5$ means $P(0) > 2 - 20\alpha$. From the phase line, this means $P$ is initially decreasing

(c) Suppose now that $\alpha > \frac{1}{10}$.

i. Find all equilibria and sketch the phase line. Include only the relevant range $P > 0$.

5




Now the point $2-20\alpha < 0$ (because $\alpha > \frac{1}{10}$), so there is only one interesting equilibrium for us: $P=0$. Moreover, $P(1-10\alpha - \frac{P}{2}) < 0$ for all $P \geq 0$ since $1-10\alpha < 0$ and $-P/2 < 0$. So $\frac{dP}{dt} < 0$ for every $P > 0$.

ii. Use qualitative analysis to predict the fate of the deer population if the population is initially 5 thousand deer (i.e., $P(0) = 5$).

3

The population will decrease and die out.

(d) What's so special about the value $\alpha = \frac{1}{10}$? That is, how does the nature of the system change as $\alpha$ passes through that value?

4

$\alpha = \frac{1}{10}$ is a critical hunting threshold. If $\alpha > \frac{1}{10}$, there is too much hunting for the population to survive long-term, no matter the starting size. ~~Any~~ $\alpha < \frac{1}{10}$ is a sustainable level of hunting and the population can exist in equilibrium with hunting at this level.