Name: ____________________________________

Math 224 Exam 3
October 12, 2012

1. Consider the linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{B}\mathbf{Y} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \mathbf{Y}$.

   (a) Find the eigenvalues of B.

   (b) Find the corresponding eigenvectors.

(c) What form does the general solution to this system have?

(d) Solve the initial value problem $\dfrac{d\mathbf{Y}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \mathbf{Y}$ and $\mathbf{Y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(e) What is the long-term behavior of your solution as $t \to \infty$? What about $t \to -\infty$?

(f) Sketch the phase plane for this system. Make sure to include any straight-line solutions, indicate direction of solution curves in time, and include the solution curve you found above to the initial value problem.

2. (a) What does it mean to say that λ is an eigenvalue of the 2×2 matrix $\mathbf{A}$ with eigenvector $\mathbf{V}$?

(b) Suppose that λ is an eigenvalue of the 2×2 matrix $\mathbf{A}$ with eigenvector $\mathbf{V}$. Show that $\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}$ is a solution to the linear system $\dfrac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$.

3. Recall the SIR Model of an epidemic:

$$\frac{dS}{dt} = -\alpha IS \qquad \frac{dI}{dt} = \alpha SI - \beta I \qquad \frac{dR}{dt} = \beta I,$$

where S is the portion of the population that is susceptible, I is the portion infected, and R is the portion "recovered"; i.e., not infected or susceptible.

(a) Suppose a fraction v of the population is vaccinated when the first case of a disease is discovered. What initial conditions would reflect this?

(b) What initial conditions (in terms of α and β) guarantee that at time 0, $\frac{dI}{dt} < 0$?

(c) Given α and β, what proportion of the population should be vaccinated to prevent an epidemic?

4. Recall the undamped harmonic oscillator $m\dfrac{d^2y}{dt^2} = -ky$.

(a) Verify that $y(t) = \sin\left(t\sqrt{\frac{k}{m}}\right)$ and $y(t) = \cos\left(t\sqrt{\frac{k}{m}}\right)$ are solutions to this differential equation.

(b) Solve the initial value problem $m\dfrac{d^2y}{dt^2} = -ky$ with $y(0) = -1$, $y'(0) = \sqrt{\frac{k}{m}}$.

(c) Sketch your solution to the previous part. What is its long-term behavior?
(Suggestion: plot the points $y(t)$ for $t = 0, \frac{\pi}{2}\sqrt{\frac{m}{k}}, \frac{3\pi}{4}\sqrt{\frac{m}{k}}, \pi\sqrt{\frac{m}{k}}, \frac{3\pi}{2}\sqrt{\frac{m}{k}}, \frac{7\pi}{4}\sqrt{\frac{m}{k}}, 2\pi\sqrt{\frac{m}{k}}$.)