Name: Solutions

Math 224 Exam 4
October 31, 2012

1. Consider the linear system $\dfrac{d\mathbf{Y}}{dt} = \mathbf{B}\mathbf{Y} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{Y}$.

   (a) Find the eigenvalues of B.

7 $\det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \boxed{1 \pm i}$$

   (b) Find the corresponding eigenvectors.

7 $\lambda = 1+i$: $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: take $\begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda = 1-i$: $\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: take $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

(c) What form does the general solution to this system have?

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There is a (complex-valued) solution given by:

$$Y(t) = e^{(1+i)t}\begin{pmatrix}1\\ i\end{pmatrix} = e^t[\cos t + i\sin t]\left[\begin{pmatrix}1\\0\end{pmatrix}+i\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$= e^t\left[\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + i\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\right]$$

$$\Rightarrow Y_{gen'l}(t) = k_1 e^t\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + k_2 e^t\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

(d) Solve the initial value problem $\frac{d\mathbf{Y}}{dt} = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\mathbf{Y}$ and $\mathbf{Y}(0) = \begin{bmatrix}2\\1\end{bmatrix}$.

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$$Y_{gen'l}(0) = k_1\begin{pmatrix}1\\0\end{pmatrix} + k_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}k_1\\k_2\end{pmatrix}.$$

We want $\begin{pmatrix}k_1\\k_2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$, so our solution is:

$$Y(t) = 2e^t\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + e^t\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

(e) What is the long-term behavior of your solution as $t \to \infty$? What about $t \to -\infty$?

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* Note: the period of the oscillations is 2π.

Since the real part of the eigenvalues is positive, the solution is a spiral source: As $t \to \infty$, the solution spirals away from 0 (clockwise, as can be seen by computing $\frac{dY}{dt}(0)$ for our solution above). As $t \to -\infty$, the solution spirals toward the origin (counter-clockwise).

(f) Sketch the phase plane for this system. Make sure to include the solution curve you found above to the initial value problem and to indicate direction of solution curves in time.

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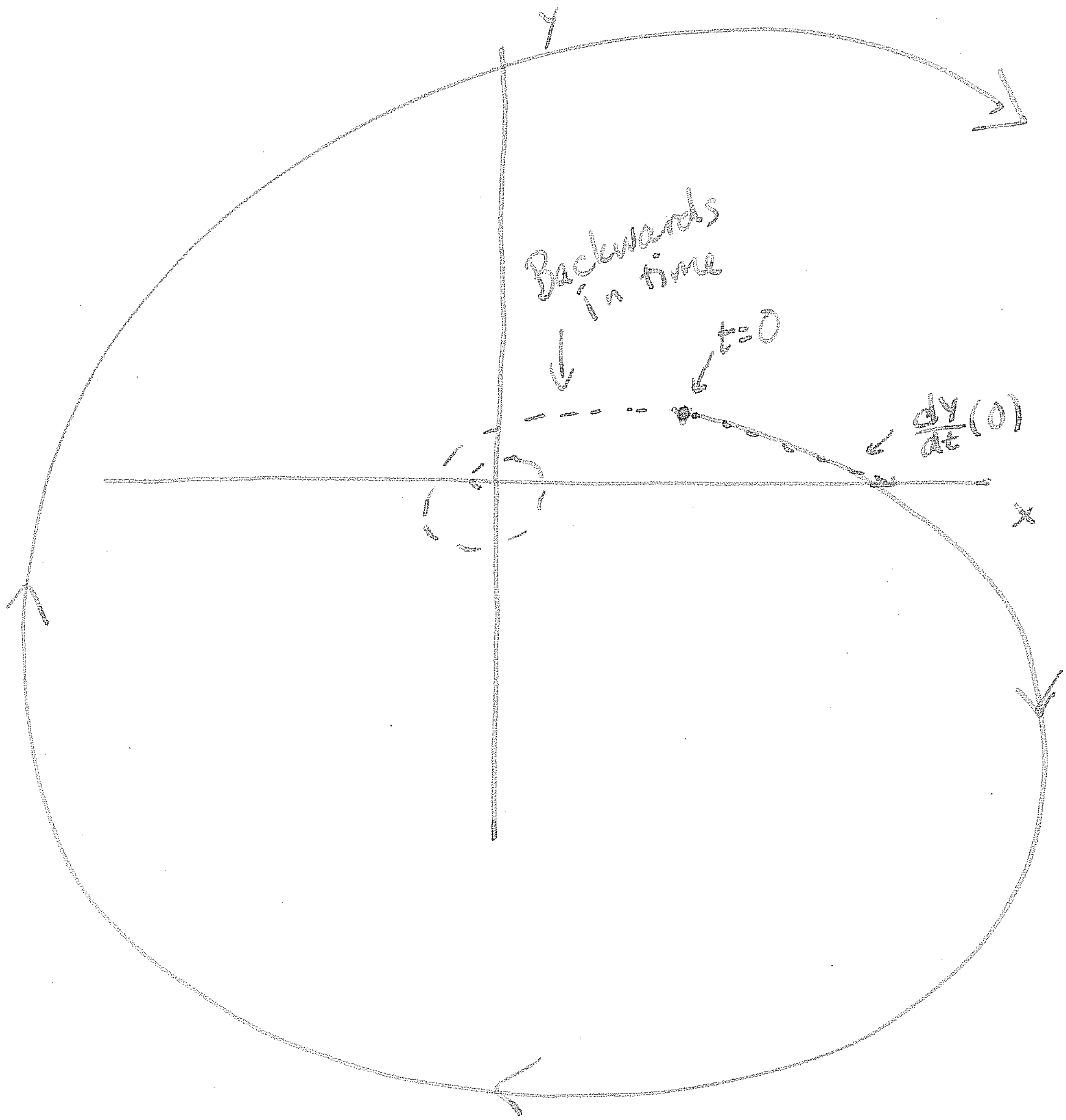

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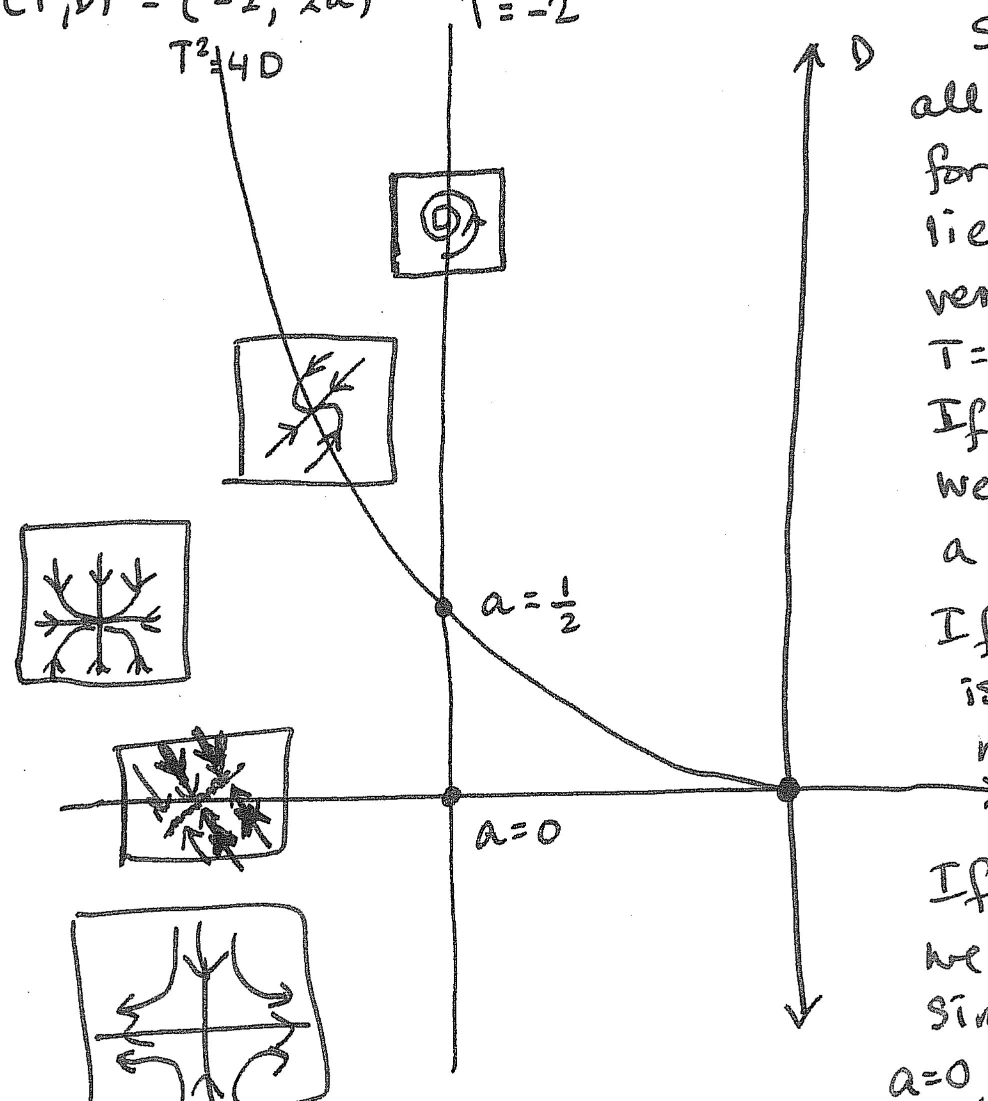
2. Consider the one-parameter family of systems

$$\frac{dY}{dt} = \begin{bmatrix} -2 & a \\ -2 & 0 \end{bmatrix} Y.$$

Sketch the relevant part of the trace-determinant plane and the curve in it that corresponds to the above systems. Describe the qualitative behavior of the system for all values of a.

$\mathrm{Tr}\begin{bmatrix} -2 & a \\ -2 & 0 \end{bmatrix} = -2$; $\det\begin{bmatrix} -2 & a \\ -2 & 0 \end{bmatrix} = 2a$

$\Rightarrow (T, D) = (-2, 2a)$



Since T=-2, all points for this system lie on the vertical line T=-2.

If $a > \frac{1}{2}$, we have a spiral sink.

If $a = 1/2$, there is a repeated negative eigenvalue.

If $a \in (0, \frac{1}{2})$, we have a sink. When a=0, one of the eigenvalues is 0 and one is negative.

For a<0, we have a saddle.

3. Consider the following model for the profits of Paul's and Bob's cafés and Glen's ice-cream store. Let $x(t)$ denote Paul's profits, $y(t)$ denote Bobs profits, and $z(t)$ denote Glen's profits, and suppose that they satisfy the system of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= -y+z\\ \frac{dy}{dt} &= -x+z\\ \frac{dz}{dt} &= z.\end{aligned}$$

(a) i. If Glen makes a profit, does this help or hurt Paul's and Bob's profits?

5 If Glen makes a profit, then $z > 0$, which increases $\frac{dx}{dt}$ and $\frac{dy}{dt}$ $\Rightarrow$ it helps both Paul and Bob.

ii. If Paul and Bob are both making profits, does this help or hurt Glen's profit?

5 Since $\frac{dz}{dt} = z$, Glen's profits are unaffected by Bob's or Paul's - they neither help nor hurt.

(b) Write the system in matrix form, find the eigenvalues, and identify the type of system.

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$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\det \begin{bmatrix} -\lambda & -1 & 1 \\ -1 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = \lambda^2(1-\lambda) + (-1)(1-\lambda) + 1(0)$$

$$= (1-\lambda)(\lambda^2-1)$$

$$= (1-\lambda)(\lambda+1)(\lambda-1)$$

Eigenvalues: 1, 1, −1 : the system is a saddle.

(c) Suppose that at time $t = 0$, $x(0)$ is small but positive, and $y(0) = z(0) = 0$. What happens to all the stores' profits? (*Hint*: think first about z, then think about x and y).

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$\frac{dz}{dt} = z$ and $z(0) = 0 \Rightarrow z(t) = 0$ for all t.

$\Rightarrow$ The system is really the 2-D system

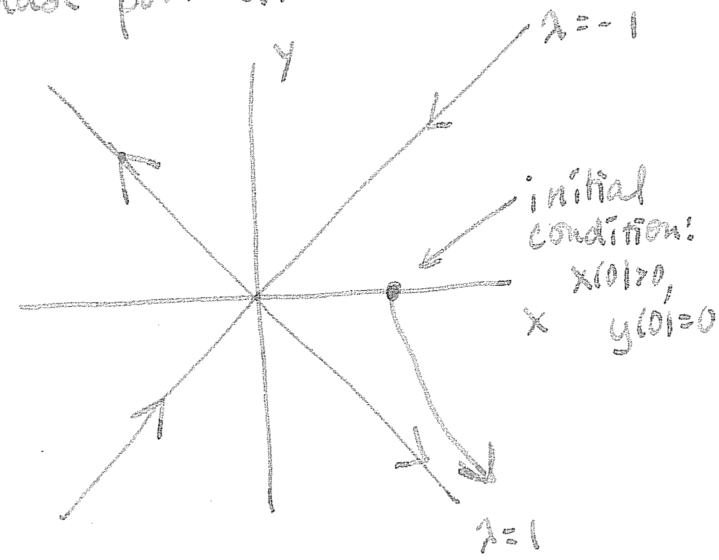
$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues: $\det \begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$: evs are ± 1.

$\lambda = 1$ $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: take $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as eigenvector

$\lambda = -1$ $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: take $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as eigenvector

Phase portrait:



From the phase portrait, we see that $x(t) \to \infty$ as $t \to \infty$ (so Paul's profits keep increasing) and $y(t) \to -\infty$ as $t \to \infty$ (so Bob loses everything).

As noted above, Glen remains at equilibrium,