Name: Solutions

Math 224 Exam 5
November 14, 2012

1. Consider the undamped, forced harmonic oscillator modeled by

$$\frac{d^2y}{dt^2} + 4y = 2\cos(\omega t).$$

(a) If $\omega \neq 2$, find the general solution to the equation above.

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The homogeneous part is $y'' + 4y = 0$, which has characteristic polynomial $\lambda^2 + 4 = (\lambda+2i)(\lambda-2i)$ $\Rightarrow$ the eigenvalues are $\pm 2i$, and the general solution is $y_{gen,h}(t) = k_1\cos(2t) + k_2\sin(2t.)$

To find a particular solution, we consider $y'' + 4y = 2e^{i\omega t}$, find a particular solution to this equation, & take the real part. Trying $y_{p,c}(t) = ke^{i\omega t}$, $y_{p,c}'(t) = ik\omega e^{i\omega t}$, $y_{p,c}'' = -\omega^2 ke^{i\omega t}$

$\Rightarrow y_{p,c}'' + 4y_{p,c} = ke^{i\omega t}[-\omega^2+4]$ $\Rightarrow$ take $k = \frac{2}{4-\omega^2}$. $\Rightarrow$ $\boxed{y_{gen}(t) = k_1\cos(2t) + k_2\sin(2t) + \frac{2}{4-\omega^2}\cos(\omega t)}$

(b) If $\omega = 3$, give the solution to the initial value problem

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$$\frac{d^2y}{dt^2} + 4y = 2\cos(\omega t), \qquad y(0) = y'(0) = 0.$$

If $\omega = 3$, $y_{gen}(t) = k_1\cos(2t) + k_2\sin(2t) - \frac{2}{5}\cos(3t)$

and $y_{gen}'(t) = -2k_1\sin(2t) + 2k_2\cos(2t) + \frac{6}{5}\sin(3t)$

$\Rightarrow y_{gen}(0) = k_1 - 2/5$ and $y_{gen}'(0) = 2k_2$

So for $y_{gen}(0) = y_{gen}'(0) = 0$, we must take $k_2 = 0$, $k_1 = 2/5$

$\Rightarrow \boxed{y(t) = \frac{2}{5}[\cos(2t) - \cos(3t)]}.$

(c) Now suppose $\omega = 2$. Find the solution to the initial value problem

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$$\frac{d^2y}{dt^2} + 4y = 2\cos(\omega t), \qquad y(0) = y'(0) = 0.$$

The general solution to the homogeneous part is still $k_1 \cos(2t) + k_2 \sin(2t)$. Since $\omega = 2$, our guess for a particular solution to $y'' + 4y = 2e^{i2t}$ is $y_{P,c}(t) = kte^{2it}$, so

$y_{P,c}'(t) = ke^{2it} + 2ikte^{2it}$, $y_{P,c}''(t) = 4kie^{2it} - 4kte^{2it}$

$\Rightarrow y_{P,c}'' + 4y_{P,c} = kte^{2it}[-4+4] + ke^{2it}(4i)$,

so we take $k = \frac{1}{2i} = \frac{-i}{2}$, and so

$y_p(t) = \mathrm{Re}\left[-\frac{i}{2}te^{2it}\right] = \frac{t}{2}\sin(2t)$

$\Rightarrow y_{gen}(t) = k_1\cos(2t) + k_2\sin(2t) + \frac{t}{2}\sin(2t)$.

(and $y_{gen}'(t) = -2k_1\sin(2t) + 2k_2\cos(2t) + \frac{1}{2}\sin(2t) + t\cos(2t)$)

$\Rightarrow y_{gen}(0) = k_1$, $y_{gen}'(0) = 2k_2 \Rightarrow k_1 = k_2 = 0$, and $\boxed{y(t) = \frac{t}{2}\sin(2t)}$

(d) Describe the long-term behavior of your solution above, and give a rough sketch.

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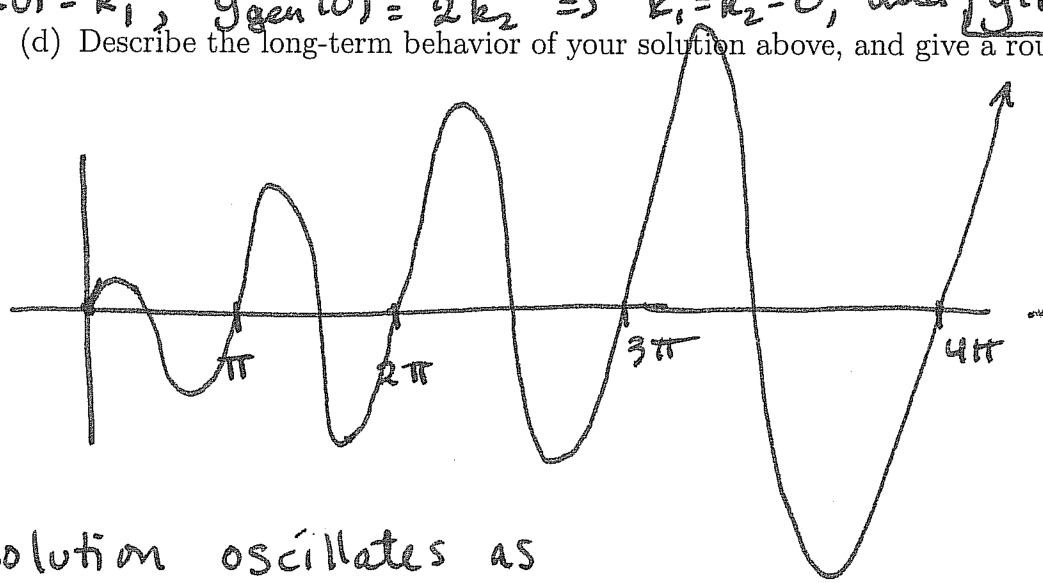


The solution oscillates as a sine wave with period π; the amplitude grows linearly (this is because we are at resonance

2. Suppose that countries X and Y are engaged in an arms race, and let $x(t)$ and $y(t)$ denote the respective sizes of the countries' stockpiles. Suppose the situation is modeled by the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= h(x,y) \\ \frac{dy}{dt} &= k(x,y). \end{aligned}$$

Suppose we have observed the following features in the rate of arms manufacture:

- If one county's stockpile in not changing, and the other increases its stockpile, it also slows its production rate.
- If either country increases its stockpile, the other responds by increasing its production rate.

(a) What do the assumptions imply about $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial y}$, $\frac{\partial k}{\partial x}$, and $\frac{\partial k}{\partial y}$?

Since x going up causes $\frac{dx}{dt}$ to go down, $\frac{\partial h}{\partial x} < 0$; similarly, $\frac{\partial k}{\partial y} < 0$.

Since x going up causes $\frac{dy}{dt}$ to go up, $\frac{\partial k}{\partial x} > 0$; similarly, $\frac{\partial h}{\partial y} > 0$.

(b) What types of equilibria are possible for this system?

At an equilibrium, the Jacobian of the system has the form $J = \begin{bmatrix} -a & b \\ c & -d \end{bmatrix}$ with $a, b, c, d > 0$.

$\Rightarrow Tr(J) = -(a+d)$, $Det(J) = ad - bc$. So since $Tr(J) < 0$, any equilibrium is a sink of some kind or a saddle.

Actually, the eigenvalues are $\lambda = \dfrac{-(a+d) \pm \sqrt{(a+d)^2 - 4ad + 4bc}}{2}$

$= \dfrac{-(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$,

Which are both real $\Rightarrow$ no spiral sinks only real sinks & saddle

(c) Suppose that both countries don't slow their production much when they add to their stockpiles, but that their reactions to enemy stockpile growth are extreme (that is, they increase production rapidly in response to enemy stockpile growth). If there are any equilibria in the system, what sort would you expect?

The assumption is that a and d are small but b and c are large

$\Rightarrow$ $\mathrm{Det}(J) = ad - bc < 0$ $\Rightarrow$ the system has only saddle equilibria.

(d) (*Extra Credit*) Can you tell what is likely to happen in the arms race?

We know any equilibrium is a saddle; moreover, the eigenvectors can be computed:

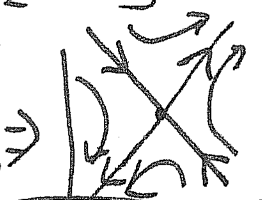
$\frac{-(a+d)+\sqrt{(a-d)^2+4bc}}{2}$: (the positive eigenvalue)

$$\begin{bmatrix} \frac{d-a-\sqrt{(a-d)^2+4bc}}{2} & b \\ c & \frac{a-d-\sqrt{(a-d)^2+4bc}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\frac{-(a+d)-\sqrt{(a-d)^2+4bc}}{2}$ (the negative eigenvalue)

$$\begin{bmatrix} \frac{d-a+\sqrt{(a-d)^2+4bc}}{2} & b \\ c & \frac{a-d+\sqrt{(a-d)^2+4bc}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

take $\begin{bmatrix} b \\ \frac{a-d+\sqrt{(a-d)^2+4bc}}{2} \end{bmatrix}$ (positive slope) $\Rightarrow$



phase plane

$\Leftarrow$ take $\begin{bmatrix} b \\ \frac{a-d-\sqrt{(a-d)^2+4bc}}{2} \end{bmatrix}$ (negative slope)

- Either both stockpiles grow exponentially

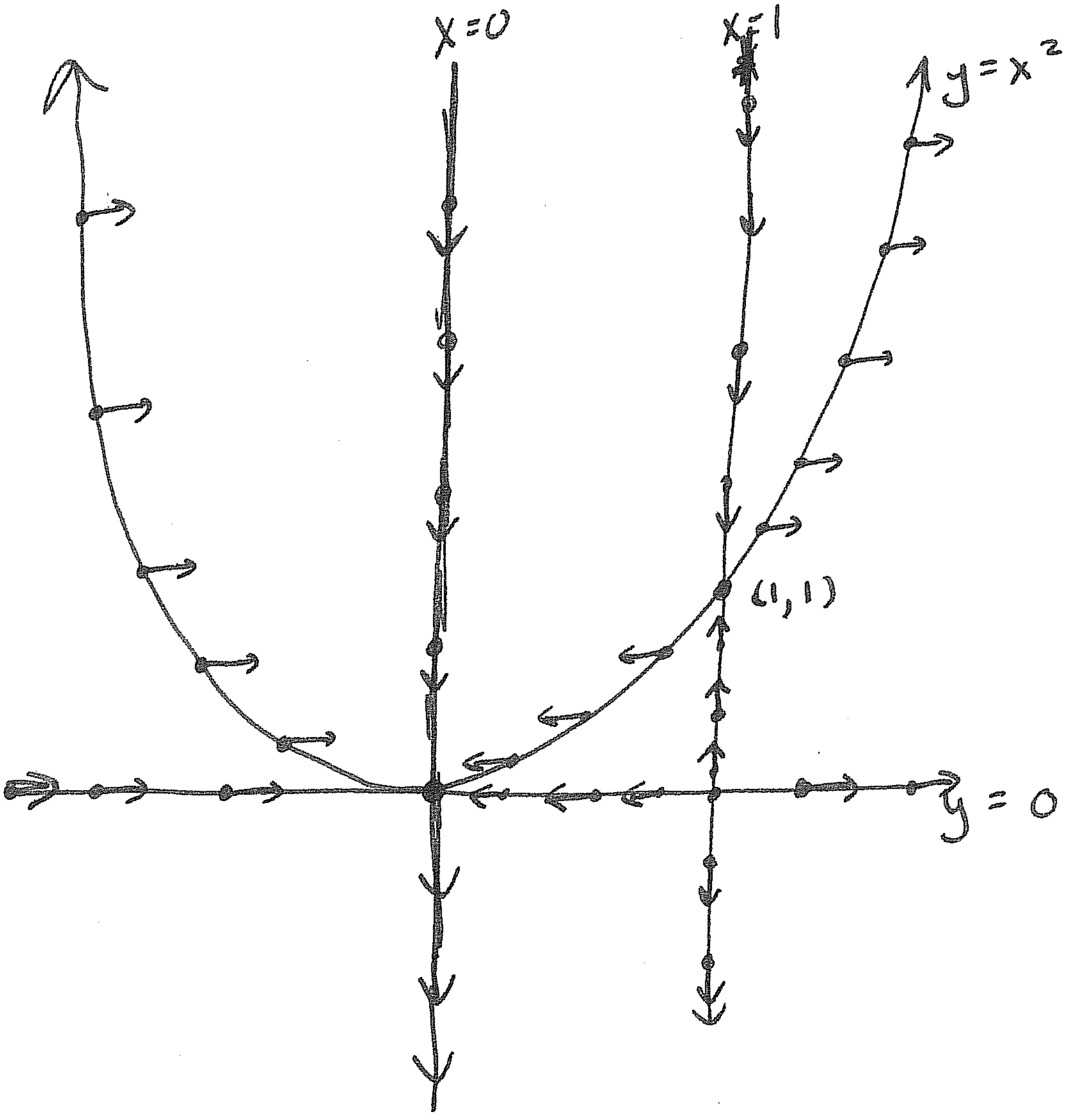
3. Consider the non-linear system

$$\frac{dx}{dt} = x(x-1)$$
$$\frac{dy}{dt} = y(x^2-y).$$

(a) Identify and sketch the nullclines (indicating the direction of the slope field along them).

x-nullclines: $x(x-1)=0 \Rightarrow \{x=0\}, \{x=1\}$

y-nullclines: $y(x^2-y)=0 \Rightarrow \{y=0\}, \{y=x^2\}$



To find directions, test values:

x=1	$\frac{dy}{dt}(1,2)=-2$
	$\frac{dy}{dt}(1,\frac{1}{2})=\frac{1}{4}$
	$\frac{dy}{dt}(1,-1)=-2$
x=0	$\frac{dy}{dt}(0,1)=-1$
	$\frac{dy}{dt}(0,-1)=-1$
$y=x^2$	$\frac{dx}{dt}(2,4)=2$
	$\frac{dx}{dt}(\frac{1}{2},\frac{1}{4})=-\frac{1}{4}$
	$\frac{dx}{dt}(-1,1)=2$
y=0	$\frac{dx}{dt}(2,0)=2$
	$\frac{dx}{dt}(\frac{1}{2},0)=-\frac{1}{4}$
	$\frac{dx}{dt}(-1,0)=2$

(b) Classify the non-trivial equilibrium (i.e., the one for which neither x nor y is 0).

There is a non-trivial equilibrium at $(1,1)$.

The linearized version of the system is

$$\frac{du}{dt} = u \qquad \text{or} \qquad \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\frac{dv}{dt} = 2u - v$$

The characteristic polynomial is

$(1-\lambda)(-1-\lambda)$, so the eigenvalues

are ± 1: the original system has

a saddle at $(1,1)$.