The spectra of powers of random unitary matrices

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March 11, 2013

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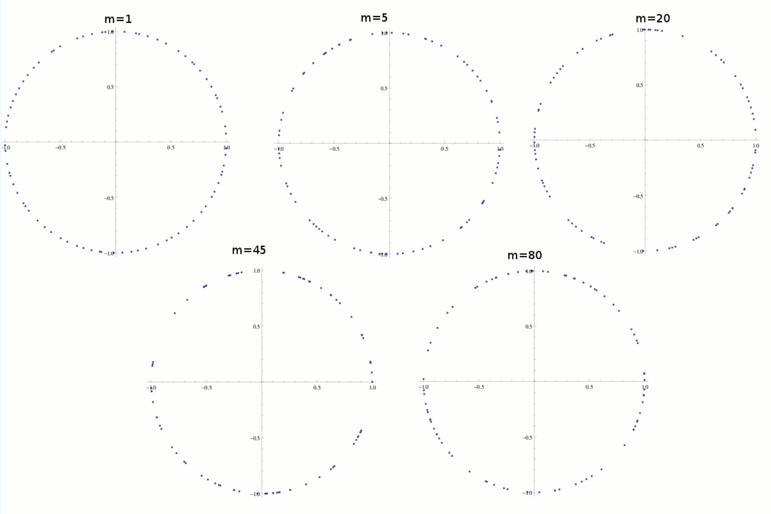
Then U^m has (random) eigenvalues $\{e^{i\theta_j}\}_{j=1}^N$.

We consider the empirical spectral measure of U^m:

$$\mu_{m,N} := \frac{1}{N} \sum_{j=1}^{N} \delta_{e^{i\theta_j}}.$$



The eigenvalues of U^m for $m = 1, 5, 20, 45, 80$, for U a realization of a random 80×80 unitary matrix.

Main results

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Theorem (E.M./M. Meckes)

Let ν denote the uniform probability measure on the circle and

$$W_p(\mu,\nu) := \inf \left\{ \left(\int |x-y|^p \, d\pi(x,y) \right)^{\frac{1}{p}} \,\middle|\, \begin{array}{l} \pi(A \times \mathbb{C}) = \mu(A) \\ \pi(\mathbb{C} \times A) = \nu(A) \end{array} \right\}.$$

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Then

- $\mathbb{E}\left[W_p(\mu_{m,N},\nu)\right] \leq \frac{Cp\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}}{N}.$

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Then

- $\mathbb{E}\left[W_p(\mu_{m,N},\nu)\right] \leq \frac{Cp\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}}{N}.$
- *For* $1 \leq p \leq 2$*,*
  $$\mathbb{P}\left[W_p(\mu_{m,N},\nu) \geq \frac{C\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}}{N} + t\right] \leq \exp\left[-\frac{N^2 t^2}{24m}\right].$$
- *For* $p > 2$*,*
  $$\mathbb{P}\left[W_p(\mu_{m,N},\nu) \geq \frac{Cp\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}}{N} + t\right] \leq \exp\left[-\frac{N^{1+\frac{2}{p}} t^2}{24m}\right].$$

Almost sure convergence

Corollary

For each N, let U_N be distributed according to uniform measure on $\mathbb{U}(N)$ and let $m_N \in \{1, \ldots, N\}$. There is a C such that, with probability 1,

$$W_p(\mu_{m_N,N}, \nu) \leq \frac{Cp\sqrt{m_N \log(N)}}{N^{\frac{1}{2}+\frac{1}{\max(2,p)}}}$$

eventually.

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Theorem (Hough/Krishnapur/Peres/Virág 2006)

Let $\mathcal{X}$ be a determinantal point process in Λ satisfying some niceness conditions. For $D \subseteq \Lambda$, let $\mathcal{N}_D$ be the number of points of $\mathcal{X}$ in D. Then

$$\mathcal{N}_D \overset{d}{=} \sum_k \xi_k,$$

where $\{\xi_k\}$ are independent Bernoulli random variables with means given explicitly in terms of the kernel of $\mathcal{X}$.

A miraculous representation of the eigenvalue counting function

That is, if $\mathcal{N}_\theta$ is the number of eigenangles of U between 0 and θ, then

$$\mathcal{N}_\theta \stackrel{d}{=} \sum_{j=1}^{N} \xi_j$$

for a collection $\{\xi_j\}_{j=1}^{N}$ of independent Bernoulli random variables.

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Theorem (Rains 2003)

Let $m \leq N$ be fixed. Then

$$[\mathbb{U}(N)]^m \overset{e.v.d.}{=} \bigoplus_{0 \leq j < m} \mathbb{U}\left(\left\lceil \frac{N-j}{m} \right\rceil\right),$$

where $\overset{e.v.d.}{=}$ denotes equality of eigenvalue distributions.

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So: if $\mathcal{N}_{m,N}(\theta)$ denotes the number of eigenangles of U^m in $[0, \theta)$, then

$$\mathcal{N}_{m,N}(\theta) \overset{d}{=} \sum_{j=1}^{N} \xi_j,$$

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- From Bernstein's inequality and the representation of $\mathcal{N}_{m,N}(\theta)$ as $\sum_{j=1}^{N} \xi_j$,

$$\mathbb{P}\left[\left|\mathcal{N}_{m,N}(\theta) - \mathbb{E}\mathcal{N}_{m,N}(\theta)\right| > t\right] \leq 2\exp\left[-\min\left\{\frac{t^2}{4\sigma^2}, \frac{t}{2}\right\}\right],$$

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- $\operatorname{Var}\left[\mathcal{N}_{1,N}(\theta)\right] \leq \log(N) + 1$ (e.g., via explicit computation with the kernel of the determinantal point process), and so

$$\operatorname{Var}\left(\mathcal{N}_{m,N}(\theta)\right) = \sum_{0 \leq j < m} \operatorname{Var}\left(\mathcal{N}_{1,\left\lceil\frac{N-j}{m}\right\rceil}(\theta)\right) \leq m\left(\log\left(\frac{N}{m}\right) + 1\right).$$

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The previous slide leads easily to the estimate

$$\mathbb{P}\left[\left|\theta_j - \frac{2\pi j}{N}\right| > \frac{4\pi t}{N}\right] \leq 4 \exp\left[-\min\left\{\frac{t^2}{m\left(\log\left(\frac{N}{m}\right)+1\right)}, t\right\}\right],$$

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If $\nu_N := \frac{1}{N}\sum_{j=1}^N \delta_{\exp\left(i\frac{2\pi j}{N}\right)}$, then $W_p(\nu_N, \nu) \leq \frac{\pi}{N}$ and

$$\mathbb{E} W_p^p(\mu_{m,N}, \nu_N) \leq \frac{1}{N}\sum_{j=1}^N \mathbb{E}\left|\theta_j - \frac{2\pi j}{N}\right|^p$$

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$$\begin{aligned}
\mathbb{E} W_p^p(\mu_{m,N}, \nu_N) &\leq \frac{1}{N}\sum_{j=1}^{N} \mathbb{E}\left|\theta_j - \frac{2\pi j}{N}\right|^p \\
&\leq 8\Gamma(p+1)\left(\frac{4\pi\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}}{N}\right)^p.
\end{aligned}$$

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- By Rains' theorem, it is distributionally the same as $F_p(U_1, \ldots, U_m) = \left(\frac{1}{m}\sum_{j=1}^{m} \mu_{U_j}, \nu\right)$.

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- By Rains' theorem, it is distributionally the same as $F_p(U_1, \ldots, U_m) = \left(\frac{1}{m}\sum_{j=1}^m \mu_{U_j}, \nu\right)$.
- $F_p(U_1, \ldots, U_m)$ is Lipschitz (w.r.t. the L_2 sum of the Euclidean metrics) with Lipschitz constant $N^{-\frac{1}{\max(p,2)}}$.
- If we had a general concentration phenomenon on $\bigoplus_{0\le j<m} \mathbb{U}\left(\left\lceil \frac{N-j}{m} \right\rceil\right)$, concentration of $W_p\left(\mu_{U^m}, \nu\right)$ would follow.

The sharp LSI on $\mathbb{U}(N)$

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Lemma

If

- θ *is uniform in* $\left[0, \frac{2\pi}{N}\right]$
- V *is uniform in* $\mathbb{SU}(N)$,
- θ *and* V *are independent,*

then $e^{i\theta} V$ *is distributed uniformly in* $\mathbb{U}(N)$.

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Proof.

Let K be uniform in $\{1, \ldots, N\}$, X uniform in $(0, 1)$ and V uniform in $\mathbb{SU}(N)$.

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Proof.

Let K be uniform in $\{1, \ldots, N\}$, X uniform in $(0, 1)$ and V uniform in $\mathbb{SU}(N)$. Look at

$$e^{\frac{2\pi iX}{N}} e^{\frac{2\pi iK}{N}} V.$$

$\square$

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Fact 4: The function

$$\begin{aligned} F: \quad \left[0, \frac{\pi\sqrt{2}}{\sqrt{N}}\right] \times \mathbb{SU}(N) &\to \mathbb{U}(N) \\ (t, V) &\mapsto e^{\frac{\sqrt{2}it}{\sqrt{N}}} V \end{aligned}$$

is $\sqrt{3}$-Lipschitz and pushes forward the product of uniform measures on $\left[0, \frac{\pi\sqrt{2}}{\sqrt{N}}\right]$ and $\mathbb{SU}(N)$ to uniform measure on $\mathbb{U}(N)$.

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One more application of tensorization gives that $\bigoplus_{0\leq j<m} \mathbb{U}\left(\left\lceil \frac{N-j}{m} \right\rceil\right)$ satisfies a log-Sobolev inequality with constant $6\left\lceil \frac{N}{m} \right\rceil$.

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Via the Herbst argument, this leads to:

$$\mathbb{P}\Big[F(U_1, \ldots, U_m) \geq \mathbb{E}F(U_1, \ldots, U_m) + t\Big] \leq \exp\left[-\frac{Nt^2}{12L^2}\right],$$

where F is L-Lipschitz.