Conformal maps, Green’s function and Poisson kernel.

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1 The reflection method for half-space problems

To get Green’s functions (for Laplace; heat; wave) on the half-space $\mathbb{R}^{n+1}_+ = \{z = (x, y) : y > 0\}$ we reflect the free-space source $K_0(z - w)$ about the boundary plane

$$w = (\xi, \eta) \rightarrow w^* = (\xi, \eta)$$

and take a suitable combination of $K(z, w)$ and $K(z, w^*)$, i.e. use odd/even extensions of $K_0$ in the vertical variable $y$,

$$K(z, w) = K_0(z - w) - K_0(z - w^*) \quad \text{Dirichlet}$$
$$K(z, w) = K_0(z - w) + K_0(z - w^*) \quad \text{Neumann}$$

In a similar way one handles the heat and wave propagators, e.g.

$$G(z, w; t) = G_0(z - w; t) - G_0(z - w^*; t)$$

in terms of the free-space Gaussian $G_0$.

The corresponding Poisson kernels are computed from $K; G; \cdots$ via standard relations, e. g.

$$P(z, w) = \partial_\eta K(z, w)|_{w \in \Gamma} = \partial_\eta K|_{\eta = 0} \quad \text{Dirichlet}$$
$$K(z, w) = K(z, w)|_\Gamma = K|_{\eta = 0} \quad \text{Neumann}$$

In particular for spherically-symmetric problem, like Laplacian/Helmholtz potentials $K_0 = K_0(r); r = |z - w|$ we get

$$P(x - \xi, y) = -\frac{2y}{r} K'_0(r); \text{with } r = \sqrt{(x - \xi)^2 + y^2}$$

In special cases 2D; 3D it yields familiar expressions, that could be also produced by the Fourier transform

$$P = \frac{y}{\pi[(x-\xi)^2+y^2]} \quad \text{2D}$$
$$P = \frac{y}{2\pi[|x-\xi|^2+y^2]^{1/2}} \quad \text{3D}$$
$$P = \frac{1}{\omega_{n-1}[|x-\xi|^2+y^2]^{n/2}} \quad \text{nD}$$

$\omega_{n-1}$-area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. 

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Exercise 1

1. Show that \( P(x - \xi, y) \to \delta(x - \xi) \), as \( y \to \infty \)

2. Derive Poisson propagators for the heat and wave problems

1.1 Examples of Green's functions for the Laplacian.

- The free space Green's function in \( \mathbb{R}^3 \) is given by the Newton-Coulomb potential (single point charge): \( K_0 = \frac{1}{4\pi r} \).

- The half-space Green's function is given by the dipole-potential - a pair of opposite point charges:

\[
K(x, y, a) = K_0(x - a, y) - K_0(x + a, y)
\]

- The quadrant Green's function is given by the quadrupole potentials,

\[
\begin{align*}
K_N(x, y, \xi, \eta) &= K_0(x - \xi, y - \eta) + K_0(x - \xi, y + \eta) + K_0(x + \xi, y - \eta) + K_0(x + \xi, y + \eta) \\
K_D(x, y, \xi, \eta) &= K_0(x - \xi, y - \eta) - K_0(x + \xi, y - \eta) - K_0(x - \xi, y + \eta) + K_0(x + \xi, y + \eta)
\end{align*}
\]

Figure 1: Source and reflected source
Figure 2: Half-space Green

Figure 3: Quadrant (Dirichlet) Green

Figure 4: Quadrant Neumann Green
2 Conformal coordinate change in Laplacian

We take map \( \phi : x \to y = \phi(x) \) in \( \mathbb{R}^n \) (or a region in \( \mathbb{R}^n \)), and denote by \( A(x) = \phi'(x) \) its Jacobian matrix, and by \( J = \det(A) \) - the Jacobian determinant. The general change of variable formula for the Laplacian has the form

\[
\Delta_y = \frac{1}{J} \nabla_y : J (TAA)^{-1} \nabla_x
\]  

- a consequence of the grad and div-transformations

\[
x \to y = \phi(x) \\
\nabla_x \to \nabla_y = (T)A^{-1} \nabla_x \\
\nabla_x \cdot \cdots \to \nabla_y \cdot \cdots = \frac{1}{J} \nabla_x (JA^{-1} \cdot \cdots)
\]

The orthogonal map \( \phi \) has orthogonal column-vectors in the Jacobian matrix

\[
A = [a_1 \bar{u}_1; a_2 \bar{u}_2; \ldots; a_n \bar{u}_n];
\]

\[
\bar{u}_i \cdot \bar{u}_j = \delta_{ij}
\]

Hence diagonal product \( (TAA) = \text{diag}\{a_1^2; \ldots; a_n^2\} \) and \( J = a_1 \ldots a_n \) in (1).

The standard example are spherical coordinates in \( \mathbb{R}^2; \mathbb{R}^3 \) etc., where

\[
\phi : \begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}
\]

\[
A = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}; (TAA) = \begin{bmatrix} 1 \\ 1/r^2 \end{bmatrix}; J = r
\]

hence the resulting spherical Laplacian

\[
\frac{1}{r} \left( \partial_r \partial_r + \frac{1}{r^2} \partial_\theta \partial_\theta \right) = \frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \partial_\theta^2
\]

Conformal maps \( \phi \) have conformal Jacobian matrix \( A = \rho U \) - “scalar times orthogonal”, so they make up a subclass of orthogonal transformations, and their determinant \( J = \rho^n \). The resulting Laplacian

\[
\Delta = \frac{1}{\rho^n} \nabla \cdot \rho^{n-2} \nabla
\]

2.1 Radial conformal map and harmonic functions

We are looking for a change of variables (dependent and independent) \( u(y) \to \rho^n (u \circ \phi) = v(x) \), that would take any harmonic function \( u(x) \) into another harmonic \( v(x) \). Substitution into (2) yields,

\[
\frac{1}{\rho^n} \nabla \cdot \rho^{n-2} \nabla (\rho^n u \circ \phi) = \rho^{-2+\alpha} \left\{ \nabla^2 u + (n - 2 + 2\alpha) \frac{\nabla \rho}{\rho} \cdot \nabla u + \alpha \left( \nabla \cdot \left( \frac{\nabla \rho}{\rho} \right) + (n - 2 + \alpha) \left| \frac{\nabla \rho}{\rho} \right|^2 \right) u \right\}
\]
To get the Laplace’s equation for $v$ the first and 0-th order terms of (3) must vanish. Hence, $\alpha = -\frac{n-2}{2}$ and the conformal factor $\rho$ solves a nonlinear PDE,

$$\nabla \cdot \left( \frac{\nabla \rho}{\rho} \right) + (n-2 + \alpha) \left| \frac{\nabla \rho}{\rho} \right|^2 = 0$$

We rewrite it for the log-derivative of $\rho$ as,

$$\Delta (\ln \rho) + \frac{n-2}{2} |\nabla (\ln \rho)|^2 = 0 \quad (4)$$

The latter could be explicitly solved when $\rho$, hence $\psi = \ln \rho$ are radial functions, $\psi = \psi(r).$ Indeed, (4) becomes a radial ODE for $\psi,$

$$r^{1-n} (r^{n-1} \psi')' + \frac{n-2}{r^2} (\psi')^2 = 0 \quad (5)$$

transferred via change of variable $r^{n-1} \frac{d}{dr} = \frac{d}{dz}$ into $\psi'' + \frac{n-2}{r^2} (\psi')^2 = 0$, and solve by separation. The off-shoot is a general solution of (5) in the form

$$\psi = C_1 \left( C_2 + r^{2-n} \right)^{2/(n-2)} \quad (6)$$

A special family of solutions vanishing at $\infty$ is given by $\psi = C \frac{1}{r}$.

### 2.2 Inversion

Let us note that $\rho = \frac{1}{r^2}$ is precisely the conformal factor of the inversion map, $\phi : x \to \frac{x}{|x|^2}$. Indeed, its Jacobian-matrix

$$A = \phi' = \frac{1}{|x|^2} \left\{ \delta_{ij} - 2 \frac{x_i x_j}{|x|^2} \right\} = r^{-2}U$$

where $U$ is an orthogonal reflection-matrix about the normal hyper-plane to $x$ in $\mathbb{R}^n$ (fig.5). In fact, one could show that a radial map $x \to \phi(r) x$ is conformal, iff scalar factor $\phi = \frac{1}{r^2}$, so inversion is the only possibility.

We have thus shown, that any harmonic function $u(x)$ gives rise to another harmonic function $u^*(x) = |x|^{2-n} u \left( \frac{x}{|x|^2} \right)$.

### 2.3 Green’s function in the ball

Clearly, both $u$ and $u^*$ take on the same value on the unit sphere $S = \{|x| = 1\}$, so inversion $\phi$ transforms harmonic functions $\{u\}$ in the interior of the unit ball $B = \{|x| \leq 1\}$ to harmonic functions $\{u^*\}$ in the exterior $\{|x| \geq 1\}$. Applying ?? to the Newton potential $u = K_0(|x - \xi|)$ - the free-space Green’s function, we get the requisite harmonic correction to $K_0$ in the unit ball $B$,

$$u^* = |x|^{2-n} K_0 \left( |x^* - \xi| \right); \quad x^* = \frac{x}{|x|^2} \quad (7)$$
Figure 5: Jacobian matrix of the inversion is the reflection about the normal plane to \( x \).

Hence, remembering that \( K_0 = \frac{1}{c_n |x-\xi|^n} \) (constant \( c_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \)) we get the Green’s function of the ball

\[
K(x, \xi) = K_0(|x-\xi|) - K_0(|\hat{x} - |x|\xi|) = K_0(|x-\xi|) - K_0 \left( \left| \hat{\xi} - |\xi| x \right| \right)
\]

(8)

Here \( \hat{x} = \frac{x}{|x|} \) and \( \hat{\xi} = \frac{\xi}{|\xi|} \) denote normalized (unit) vectors in the direction \( x \) and \( \xi \). It is often convenient to rewrite (8) in the polar form, i.e. variables \( r = |x|, \rho = |\xi| \) and angle \( \theta \) between \( x \) and \( \xi \) (see fig.6),

\[
K(r, \rho; \theta) = K_0 \left( \sqrt{r^2 + \rho^2 - 2r\rho \cos\theta} \right) - K_0 \left( \sqrt{1 + (r\rho)^2 - 2r\rho \cos\theta} \right)
\]

(9)

2.4 Special cases.

2.4.1 2-D case:

Here \( K_0 = \frac{1}{2\pi} \ln |z - w| \), complex variables \( z, w \in \mathbb{C} \) being used in place of \( x, \xi \). The prefactor \( |z|^n = 1 \) in (7), hence (8-9) takes the form

\[
K(z, w) = \frac{1}{2\pi} \ln \left| \frac{z - w}{1 - z\bar{w}} \right| = \frac{1}{2\pi} \ln \sqrt{\frac{r^2 + \rho^2 - 2r\rho \cos\theta}{1 + (r\rho)^2 - 2r\rho \cos\theta}}
\]

(10)

in polar coordinates \( z = re^{i\theta}, w = \rho e^{i\phi} \). Exterior disk Green’s function has the same form (10), but this time \( r, \rho \) lie outside the unit circle
Figure 6:

Figure 7: Interior and exterior disk Green’s function
2.4.2 3-D case:

Here free-space \( K_0 = \frac{1}{4\pi|x-\xi|} \), hence

\[
K(x; \xi) = \frac{1}{4\pi} \left\{ \frac{1}{|x-\xi|} - \frac{1}{||(\hat{x} - |x|\xi)||} \right\} = \frac{1}{4\pi} \left\{ \frac{1}{\sqrt{r^2 + \rho^2 - 2r\rho \cos \theta}} - \frac{1}{\sqrt{1 + (rp)^2 - 2r\rho \cos \theta}} \right\}
\]

2.5 Poisson kernels

For the Dirichlet boundary condition the Poisson kernel is given by normal derivative of the Green’s function, \( P(x, \xi) = n_\xi \cdot \nabla_\xi K = \frac{\partial}{\partial \rho} K \).

We observe that on the boundary \( |\xi| = 1 \), \( n_\xi = \xi \). Hence,

\[
P(x, \xi) = \frac{(\xi - x) \cdot \xi}{c_n |\xi - x|^n} - \frac{(\xi - x^*) \cdot \xi}{c_n |x|^{n-2} |\xi - x^*|^n} \tag{11}
\]

\[
= \frac{(\xi - x) \cdot \xi}{c_n |\xi - x|^n} - \frac{|x|^2 - x \cdot \xi}{c_n |(|x|\xi - \hat{x})|^n}
\]

It remains to note that both denominators in (11) are equal, so we get Poisson kernel,

\[
P = \frac{1 - |x|^2}{c_n |\xi - x|^n} = \frac{1 - r^2}{c_n (1 + r^2 - 2r \cos \theta)^{n/2}} \tag{12}
\]

From the unit ball one can easily pass to an arbitrary radius \( a \),

\[
P = \frac{a^2 - r^2}{c_n (r^2 + a^2 - 2ra \cos \theta)^{n/2}} \tag{13}
\]

Another way to derive (12) is via differentiation of the polar form (9) in variable \( \rho \) at \( \rho = 1 \),

\[
P (r, 1; \theta) = \frac{\partial K}{\partial \rho} \bigg|_{\rho=1} = K' \left( \sqrt{1 + r^2 - 2r \cos \theta} \right) \frac{1 - r^2}{\sqrt{1 + r^2 - 2r \cos \theta}} = ...
\]

In special cases we get

<table>
<thead>
<tr>
<th>Green’s functions and Poisson kernels in 2D and 3D balls</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_0 )</td>
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<tr>
<td>---------------------------------------------------------</td>
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<tr>
<td>2D - ( \frac{1}{2\pi} \ln \rho )</td>
</tr>
<tr>
<td>---------------------------------------------------------</td>
</tr>
<tr>
<td>3D ( \frac{1}{4\pi} )</td>
</tr>
</tbody>
</table>
2.6 Other examples

2.6.1 Half-disk

Green’s function and the Poisson kernel in the upper half-disk are obtained by a combination of the reflection and inversion:

\[ K(z, \zeta) = \frac{1}{2\pi} \left\{ \ln \left| \frac{z - \zeta}{1 - z\bar{\zeta}} \right| - \ln \left| \frac{\bar{z} - \bar{\zeta}}{1 - \bar{z}\zeta} \right| \right\} \]

or in polar form \( z = re^{i\theta}, \zeta = \rho e^{i\phi} \)

\[ K = \frac{1}{2\pi} \left\{ \ln \sqrt{\frac{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}{1 + (r\rho)^2 - 2r\rho\cos(\theta - \phi)}} - \ln \sqrt{\frac{r^2 + \rho^2 - 2r\rho\cos(\theta + \phi)}{1 + (r\rho)^2 - 2r\rho\cos(\theta + \phi)}} \right\} \]

The corresponding Poisson kernel is also obtained by reflection of (12)

\[ P(r; \phi, \theta) = \frac{1}{2\pi} \left\{ \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)} - \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta + \phi)} \right\} \]

on the upper (semicircular) boundary and

\[ P(r; \rho, \theta) = \frac{r\sin\theta}{\pi} \left\{ \frac{1}{r^2 + \rho^2 - 2r\rho\cos\theta} - \frac{1}{1 + (r\rho)^2 - 2r\rho\cos\theta} \right\} \]

on the lower (flat) boundary.

2.6.2 General 2-D regions in complex plane.

By the general Riemann mapping Theorem of complex analysis any planar region \( D \subseteq \mathbb{C} \) can be conformally mapped onto any other such region, e.g. onto the unit disk \( \mathbb{D} = \{ |z| < 1 \} \) or a half-plane \( \mathbb{H} = \{ \text{Im}(z) > 0 \} \).

Example 2 Map \( \phi : z \rightarrow w = \frac{z - 1}{z + 1} \) takes a half-plane \( \{ \text{Im}(z) < 0 \} \) onto the unit disk \( \{ |w| < 1 \} \), and has a conformal factor \( \rho = |\phi'(z)| = \frac{2}{|z + 1|^2} \).
Example 3 Map \( \phi : z \rightarrow e^z \) takes a complex strip \( \{0 \leq \Im(z) \leq 2\pi\} \) onto the entire plane \( \mathbb{C} \).

Example 4 Map \( w = \cosh(z) \) takes a coordinate rectangle \( \{0 \leq x \leq a; 0 \leq y \leq 2\pi\} \) in the \( z \)-plane onto an ellipse in the \( w \)-plane (elliptical coordinates).

Conformal maps allow to transfer Green’s functions from one region to another. Namely, we let map \( z \rightarrow w = \phi(z) \) take region \( D \) (with the known Green’s function \( K_D(z, \zeta) \)) onto region \( M \), whose Green’s function \( K_M \) is to be computed. Assuming the boundary of \( D \) mapped onto the boundary of \( M \), two Laplacians are related via \( \Delta_M [u] \circ \phi = \frac{1}{J} \Delta_D [u \circ \phi] \), where \( J = \rho^2 \) denotes the Jacobian determinant. It follows then that the Green’s functions of two regions are related by a change of variables

\[
K_M (w, \eta) = K_D (\phi^{-1}(w), \phi^{-1}(\eta))
\]

In the above examples map \( w = \frac{\zeta - i}{\zeta + i} \) from the half-plane \( \mathbb{H} \) onto the disk \( \mathbb{D} \). The half-space Green’s function, computed by the reflection (complex conjugation) has the form: \( K = \frac{1}{2\pi} \ln \left| \frac{\zeta - w}{\zeta + w} \right| \). Substitution of the inverse map \( z = \phi^{-1}(w) = i\frac{w + i}{w - i} \) and \( \zeta = \phi^{-1}(\eta) = \cdots \) yields

\[
K_{\mathbb{D}} (w, \eta) = \frac{1}{2\pi} \ln \left| \frac{w - \eta}{1 - w\eta} \right|
\]

that was produced earlier by the inversion method.
2.6.3 Green and Poisson kernels on strip

1. We use conformal map: $z \rightarrow e^z$ from strip $\{0 \leq \text{Im} \, z \leq \pi\}$ onto half-space $\{\text{Im} \, w \geq 0\}$. Then

$$G(z, w) = -\frac{1}{2\pi} \ln \left| \frac{e^z - e^w}{e^z - e^{-w}} \right| = -\frac{1}{2\pi} \ln \left| \frac{\sinh \left( \frac{z-w}{2} \right)}{\sinh \left( \frac{z+w}{2} \right)} \right|$$

$$= -\frac{1}{4\pi} \ln \left\{ \frac{\cosh \left( \frac{x-\xi}{2} \right) - \cos \left( \frac{y-\eta}{2} \right)}{\cosh \left( \frac{x+\xi}{2} \right) - \cos \left( \frac{y+\eta}{2} \right)} \right\}$$

$$P(z, \xi) = \partial_\nu G(z, w) \big|_{\nu=0} = \frac{1}{2\pi} \sin y \cosh (x-\xi) - \cos (y)$$

Function $P(x, y)$ along with its contour map near zero, in strip $0 < y < \pi$

2.6.4 Problems

1. Show that complex map $w = \cosh(z)$ takes strip $\mathbb{P} = \{0 \leq \text{Im}(z) \leq 2\pi\}$ onto the upper half-plane $\mathbb{H}$, so that the boundaries of $\mathbb{P}: \{\text{Im}(z) = 0\}$ and $\{\text{Im}(z) = 2\pi\}$ go into the boundary of $\mathbb{H}$. Use this map and the known Green’s function of $\mathbb{H}$ to construct the Green’s function and the Poisson kernel of $\mathbb{P}$.

2. Do the same exercise with the fractional power map $\phi(z) = z^{\alpha/\pi}$ that takes the upper half-plane onto a sector of angle $\alpha$, and derive the Green’s function and the Poisson kernel of the sector.

3 Applications to fluid flows

A 2D ideal (incompressible, inviscid) Euler flow is described by its stream-field $\psi$. When viewed as a function of complex variable $z = x + iy$, $\psi(z, \bar{z})$, the
Eulerian velocity \( \mathbf{u} = (-\psi_y, \psi_x) \) is given by a complex derivative \( \mathbf{u} = i\partial_z \psi \). Irrational flow \( \nabla \times \mathbf{u} = 0 \) corresponds to harmonic stream-field: \( \Delta \psi = 0 \).

Equivalently \( \mathbf{u} \) has velocity potential \( \mathbf{u} = \nabla \phi \). In fact, it could be described by a complex analytic potential \( w(z) = \phi + i \psi \), and complex velocity \( w'(z) = u - iv \). Indeed, the relations \( \mathbf{u} = \nabla \psi^+ \) are nothing but Cauchy-Riemann equations for \( w \).

Let us also remark that steady Euler flow obeys the Bernoulli relation:

\[
\frac{\rho}{2} |\mathbf{u}|^2 + p = \text{Const} \quad \text{the hydrostatic pressure}
\]

where \( \rho \) an \( p \) are fluid density and pressure. The latter follows from the momentum conservation

\[
\mathbf{u} \cdot \nabla \left( \frac{\rho}{2} |\mathbf{u}|^2 + p \right) - \mathbf{u} \times \omega = 0
\]

where \( \omega = \nabla \times \mathbf{u} \) denotes vorticity of \( \mathbf{u} \). We consider a few examples and basic problems of ideal fluid flows.

### 3.1 Potential flow passed an obstacle.

We shall study an incompressible potential flow passed an obstacle \( D \), e.g. a cylinder, or sphere of radius \( a \). It produced by a moving parallel flow passed the body, or the motion of the body in a quiescent fluid. In either case potential, laminar flow has stream-field \( \psi \) and potential \( \phi \) - both harmonic functions in the exterior of \( D \), \( \Delta \psi = \Delta \phi = 0 \) in \( \mathbb{R}^n \setminus D; \quad n = 2, 3 \).

Depending on the reference frame one either fixes \( \Gamma \) and certain asymptotic flow at \( \infty \) e.g. \( \mathbf{u} \approx (U, 0) \) - (body frame), or allows moving \( \Gamma \) and zero \( \infty \)-flow (fluid frame). We shall use the latter as it permits non-stationary motion of \( D \).

The body velocity \( \mathbf{U} \) creates a boundary potential \( \phi_0 = -\mathbf{U} \cdot \mathbf{r} |\Gamma \) , to maintain the tangential flow along the boundary, i.e. \( \psi |\Gamma \) = Const. The latter is extended through the entire \( \mathbb{R}^n \setminus D \) via the exterior Poisson kernel \( \phi = P_\Gamma [\phi_0] \).

Any harmonic function \( \phi \) could be expanded at \( \infty \) as

\[
\phi = \frac{A_0}{r} + \frac{A_1 \cdot \mathbf{r}}{r^3} + \ldots + \frac{A_n (\mathbf{r})}{r^{n+1}} + \ldots
\]

with spherical harmonics \( A_n (\mathbf{r}) \) of degree \( n \). For velocity potential \( \phi \) coefficient \( A_0 = 0 \), since the net outward flux (at large \( r \)) must be 0 (no sources).

Clearly, potential \( \phi \), and velocity \( \mathbf{u} = \nabla \phi \) depend linearly on \( \mathbf{U} \)

\[
\phi = A (\mathbf{r}) \cdot \mathbf{U} \quad A_i = P_\Gamma [x_i]
\]

the components \( A_i \) represent harmonic extensions of linear coordinate functions on \( \Gamma \).

To get the fluid reaction force on the body we compute the total (kinetic) energy of the fluid

\[
E = \frac{\rho}{2} \int \int |\mathbf{u}|^2 = \frac{\rho}{2} \sum m_{ij} U_i U_j
\]
\( \rho \) - fluid density. The coefficients of the quadratic form \( E = E(U) \) make up the so-called virtual mass-tensor

\[
m_{ij} = \rho \int \int_{\mathbb{R}^n \setminus D} \nabla A_i (\bar{r}) \cdot \nabla A_j (\bar{r}) = \rho \oint_{\Gamma} A_i (\partial_n A_j)
\]

\[
= \rho \oint_{\Gamma} x_i \partial_n P_1 [x_j] = \rho \oint_{\Gamma} x_j \partial_n P_1 [x_i]
\]

Definition (18) resembles the inertia tensor for a rotating solid of mass-density \( \frac{\rho}{r} \) distributed over surface \( \Gamma \). It’s not however, equal \( \frac{\rho}{r} x_j x_i r \), as Poisson makes important difference. Though \( A_i \) coincides with a linear function \( x_i |_{\Gamma} \), its gradient \( \nabla A_i \) and normal derivative \( \partial_n A_j \) on \( \Gamma \) are not \( (\cos \theta, \sin \theta) \) as opposed to \( \nabla x |_{\Gamma} = (\cos \theta, 0) \). Let us also remark that tensor \( m_{ij} \) is a geometric invariant of surface \( \Gamma \), independent of its position in \( \mathbb{R}^n \), as one could easily verify, using translational/rotational symmetries of solution (16), and volume-integral\(^1\) form (18).

In a similar fashion one computes the total momentum of the induced flow

\[
\vec{p} = \rho \int \int_{\mathbb{R}^n \setminus D} u = \rho \oint_{\Gamma} \left( \sum_j U_j x_j \right) = \left( \sum \cdots m_{ij} U_j \right)
\]

The rate of the change of the momentum gives the fluid reaction force, exerted on the body. So the resulting equation of motion take the

\[
M \frac{dU}{dt} + \frac{dp}{dt} = F - \text{external force}
\]

where \( M = \int_D \rho_0 \) is the body mass. Remembering the exact form of fluid momentum (19) \( \sum_j (M \delta_{ij} + m_{ij}) \frac{dU_j}{dt} = F_i \). Thus the effect of fluid on the moving body is to replace the standard mass by the virtual mass-tensor in the kinetic energy. The components of the reaction force \(- \frac{\partial p}{\partial U} \) along \( U \), and its normal \( U^\perp \) gives the drag and buoyancy (lift) forces

\[
F_d = - \sum_j m_{ij} \dot{U}_i U_j; \quad F_b = - \sum_j m_{ij} \dot{U}_i U^\perp_j
\]

In the 2D-case and horizontal \( U = (U, 0) \) we get \( F_d = -m_{11} U \dot{U}; F_b = -m_{12} U \dot{U} \).

**Exercise 5** Compute virtual mass-tensor and the drag and buoyancy forces for the cylinder (disk in 2D), half-cylinder \((0 \leq \theta \leq \pi)\), and quarter-cylinder \((0 \leq \theta \leq \pi/2)\). Do the same exercise for solid sphere, hemi-sphere and quarter-sphere in 3D.

\(^1\)Observe, that coefficients \( m_{ij} \), hence the total energy of the induced flow is finite, as functions \( A_i (\bar{r}) \approx O (r^{-2}) \) decay sufficiently fast at \( \infty \).
Remark 6 Notice that the reaction force depends on acceleration \( \dot{U} \), so there is no net drag or lift in a uniform flow (d’Alembert paradox). This is not surprising, particularly for the drag-force, as its presence would require either energy dissipation or a non-zero energy flux to \( \infty \). Both are absent in the ideal fluid.

We shall demonstrate the general principles with two specific examples.

3.1.1 Potential flow passed the cylinder.

of parallel flow passed the cylinder \( r \leq a \). Its stream-field is the sum of the principal (parallel) flow-component \( \Psi = -Uy \) and a perturbation \( \psi' \), that would make the sum \( \psi = \Psi + \psi' \) -constant (e.g. 0) on the boundary. So we solve the Poisson equation for \( \psi' (r, \theta) \) in the exterior of disk with boundary value \( -Uy \)

\[
\begin{align*}
\Delta \psi' &= 0 \text{ for } r < a \\
\psi' \big|_{r=a} &= Ua \sin \theta = -\Psi \big|_{r=a}
\end{align*}
\]

The solution is obtained via exterior Poisson kernel \( P(r, \theta) = \frac{r^2-a^2}{2\pi(r^2-2ar \cos \theta+a^2)} \)

\[
\psi' = \int_0^{2\pi} P(r, \theta - \tau) Ua \sin \tau
\]

Direct evaluation of (21) using convolution identity \( e^{ik\theta} * e^{im\theta} = 2\pi \delta_{km} \) yields

\[
\psi' = \frac{Ua \sin \theta}{r} = \frac{UaUy}{r} = U \text{Im} \left( \frac{z}{a} \right)
\]

of \( \text{Im} \left( \frac{z}{a} + \frac{a}{z} \right) = 0 \) on the circle of radius \( |z| = a \). We plot stream-lines of

\[
\psi = Ua \text{Im} \left( \frac{z}{a} + \frac{a}{z} \right) = Uy \left( 1 - \frac{a^2}{r^2} \right)
\]

From the explicit \( \psi \), one computes the velocity \( u = \partial_z \psi \) and the hydrostatic pressure (14) \( p = p_\infty - \frac{1}{2} |\nabla \psi|^2 \). The reaction forces could be obtained by
evaluating pressure gradient along the circle $F = \oint_C \nabla p \, ds$. Thus we get

$$\psi(r, \theta) = U \left( r - \frac{a^2}{r} \right) \sin \theta - \text{stream}$$

$$p(r, \theta) = -\frac{1}{2} \left\{ \left( \frac{\partial \psi(r, \theta)}{\partial r} \right)^2 + r^2 \left( \frac{\partial \psi(r, \theta)}{\partial \theta} \right)^2 \right\} - \text{pressure}$$

$$f(r, \theta) = r \cos \theta \left( \frac{\partial p(r, \theta)}{\partial r} \right) - \frac{\sin \theta}{r} \left( \frac{\partial p(r, \theta)}{\partial \theta} \right) - \text{horizontal force}$$

### 3.1.2 Point-vortex flow passed cylinder

The point-vortex at $\zeta$ of strength $\Gamma$ creates a flow passed an obstacle $D$ whose stream-field is given by the exterior (Dirichlet) Green’s function

$$-\Delta \psi = \Gamma \delta(z - \zeta)$$

$$\psi|_{\partial D} = \text{Const}$$

For the cylinder $D = \{ |z| < a \}$ we have $\psi(z, \zeta) = \frac{\Gamma}{2\pi} \ln \left| \frac{a(z - \zeta)}{a^2 - z\zeta} \right|$, and the corresponding velocity

$$u = i \partial_z \psi = \frac{i \Gamma}{4\pi} \left\{ \frac{1}{z - \zeta} + \frac{\zeta}{a^2 - z\zeta} \right\} = \frac{i \Gamma}{4\pi} \frac{a^2 - |\zeta|^2}{(z - \zeta)(a^2 - z\zeta)}$$

Due to rotational symmetry we could place source on the real axis $\zeta = b > 0$ call $z = re^{i\theta}$ and compute pressure

$$p = -\frac{\Gamma^2}{2(4\pi)^2} \left| \frac{a^2 - b^2}{|r - be^{i\theta}|^2 |br - a^2e^{i\theta}|^2} \right|^2 \tag{22}$$

Clearly, the net force is exerted by the vortex on the cylinder is directed toward the vortex. Its precise form is obtained by evaluating the pressure gradient $p_x$ of (22) along the circle $f = \frac{\Gamma^2}{2(4\pi)^2} \oint_C p_x \, ds$. 

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