Random Unitary Matrices and Friends

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- The set of all $n \times n$ unitary matrices is denoted $\mathbb{U}(n)$; this set is a group and a manifold.

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- Metric Structure:
  - $\mathbb{U}(n)$ sits inside $\mathbb{C}^{n^2}$ and inherits a geodesic metric $d_g(\cdot, \cdot)$ from the Euclidean metric on $\mathbb{C}^{n^2}$.

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- Randomness:
  There is a unique translation-invariant probability measure called Haar measure on $\mathbb{U}(n)$: if U is a Haar-distributed random unitary matrix, so are AU and UA, for A a fixed unitary matrix.

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2.
   - Fill an $n \times n$ array with i.i.d. standard complex Gaussian random variables.
   - Stick the result into the *QR* algorithm; the resulting Q is Haar-distributed on $\mathbb{U}(n)$.

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- $\mathbb{O}(n)$ has two connected components: $\mathbb{SO}(n)$ ($\det(U) = 1$) and $\mathbb{SO}^-(n)$ ($\det(U) = -1$).

- There is a unique translation-invariant (Haar) probability measure on each of $\mathbb{O}(n)$, $\mathbb{SO}(n)$ and $\mathbb{SO}^-(n)$.

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- The group of $2n \times 2n$ symplectic matrices is denoted $\mathbb{Sp}(2n)$.

Concentration of measure

Theorem (G/M;B/E;L;M/M)

Let G be one of $\mathbb{SO}(n)$, $\mathbb{SO}^{-}(n)$, $\mathbb{SU}(n)$, $\mathbb{U}(n)$, $\mathbb{S}\mathrm{p}(2n)$, and let $F : G \to \mathbb{R}$ be L-Lipschitz (w.r.t. the geodesic metric or the HS-metric). Let U be distributed according to Haar measure on G. Then there are universal constants C, c such that

$$\mathbb{P}\left[\left|F(U) - \mathbb{E}F(U)\right| > Lt\right] \leq Ce^{-cnt^2},$$

for every $t > 0$.

The entries of a random orthogonal matrix

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$\Longrightarrow$ The entries $\{u_{ij}\}$ of U are

individually approximately Gaussian

if U is large.

The entries of a random orthogonal matrix

A more modern fact (Diaconis–Freedman): If X is a randomly distributed point on the sphere of radius $\sqrt{n}$ in $\mathbb{R}^n$, and Z is a standard Gaussian random vector in $\mathbb{R}^n$, then

$$d_{TV}\Big((X_1, \ldots, X_k), (Z_1, \ldots, Z_k)\Big) \leq \frac{2(k+3)}{n-k-3}.$$

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Diaconis'question: How many entries of U can be simultaneously approximated by independent Gaussians?

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Theorem (Jiang)

Let $\{U_n\}$ be a sequence of random orthogonal matrices with $U_n \in \mathbb{O}(n)$ for each n, and suppose that $p_n, q_n = o(\sqrt{n})$.

Let $\mathcal{L}(\sqrt{n}U(p_n, q_n))$ denote the joint distribution of the $p_n q_n$ entries of the top-left $p_n \times q_n$ block of $\sqrt{n}U_n$, and let $Z(p_n, q_n)$ denote a collection of $p_n q_n$ i.i.d. standard normal random variables. Then

$$\lim_{n\to\infty} d_{TV}(\mathcal{L}(\sqrt{n}U(p_n, q_n)), Z(p_n, q_n)) = 0.$$

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$$\lim_{n\to\infty} d_{TV}(\mathcal{L}(\sqrt{n}U(p_n, q_n)), Z(p_n, q_n)) = 0.$$

That is, a $p_n \times q_n$ principle submatrix can be approximated in total variation by a Gaussian random matrix, as long as $p_n, q_n \ll \sqrt{n}$.

Jiang's answer(s)

Theorem (Jiang)

For each n, let $Y_n = \left[y_{ij}\right]_{i,j=1}^{n}$ *be an* $n \times n$ *matrix of independent standard Gaussian random variables and let* $\Gamma_n = \left[\gamma_{ij}\right]_{i,j=1}^{n}$ *be the matrix obtained from* Y_n *by performing the Gram-Schmidt process; i.e.,* Γ_n *is a random orthogonal matrix. Let*

$$\epsilon_n(m) = \max_{1 \le i \le n, 1 \le j \le m} \left| \sqrt{n}\gamma_{ij} - y_{ij} \right|.$$

Then

$$\epsilon_n(m_n) \xrightarrow[n \to \infty]{\mathbb{P}} 0$$

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if and only if $m_n = o\left(\frac{n}{\log(n)}\right)$.

That is, in an "in probability" sense, $\frac{n^2}{\log(n)}$ entries of U can be simultaneously approximated by independent Gaussians.

A more geometric viewpoint

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In general, a rank k orthogonal projection of $\mathbb{O}(n)$ looks like

$$U \mapsto \big(\mathrm{Tr}(A_1 U), \ldots, \mathrm{Tr}(A_k U) \big),$$

where $A_1, \ldots, A_k$ are orthonormal matrices in $\mathbb{O}(n)$; i.e.,

$$\mathrm{Tr}(A_i A_j^T) = \delta_{ij}.$$

A more geometric viewpoint

Theorem (Chatterjee–M.)

Let $A_1, \ldots, A_k$ be orthonormal (w.r.t. the Hilbert-Schmidt inner product) in $\mathbb{O}(n)$, and let $U \in \mathbb{O}(n)$ be a random orthogonal matrix. Consider the random vector

$$X := (\mathrm{Tr}(A_1 U), \ldots, \mathrm{Tr}(A_k U)),$$

and let $Z := (Z_1, \ldots, Z_k)$ be a standard Gaussian random vector in $\mathbb{R}^k$. Then for all $n \geq 2$,

$$d_W(X, Z) \leq \frac{\sqrt{2}k}{n-1}.$$

Here, $d_W(\cdot, \cdot)$ denotes the L_1-Wasserstein distance.

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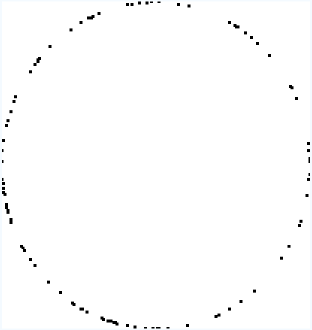
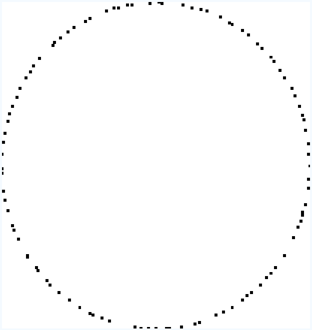
To understand the behavior of the ensemble of random eigenvalues, we consider the empirical spectral measure of U:

$$\mu_N := \frac{1}{N} \sum_{j=1}^{N} \delta_{e^{i\theta_j}}.$$



100 i.i.d. uniform random points



E. Rains

The eigenvalues of a 100×100 random unitary matrix

Diaconis/Shahshahani

Theorem (D–S)

Let $U_n \in \mathbb{U}(n)$ be a random unitary matrix, and let μ_{U_n} denote the empirical spectral measure of U_n. Let ν denote the uniform probability measure on $\mathbb{S}^1$. Then

$$\mu_{U_n} \xrightarrow{n\to\infty} \nu,$$

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- The theorem follows from explicit formulae for the mixed moments of the random vector $\big(\mathrm{Tr}(U_n), \ldots, \mathrm{Tr}(U_n^k)\big)$ for fixed k, which have been useful in many other contexts.
- They showed in particular that $\big(\mathrm{Tr}(U_n), \ldots, \mathrm{Tr}(U_n^k)\big)$ is asymptotically distributed as a standard complex Gaussian random vector.

The number of eigenvalues in an arc

Theorem (Wieand)

Let $I_j := (e^{i\alpha_j}, e^{i\beta_j})$ be intervals on $\mathbb{S}^1$ and for $U_n \in \mathbb{U}(n)$ a random unitary matrix, let

$$Y_{n,k} := \frac{\mu_{U_n}(I_k) - \mathbb{E}\mu_{U_n}(I_k)}{\frac{1}{\pi}\sqrt{\log(n)}}.$$

Then as n tends to infinity, the random vector $(Y_{n,1}, \ldots, Y_{n,k})$ converges in distribution to a jointly Gaussian random vector $(Z_1, \ldots, Z_k)$ with covariance

$$\mathrm{Cov}(Z_j, Z_k) = \begin{cases} 0, & \alpha_j, \alpha_k, \beta_j, \beta_k \text{ all distict}; \\ \frac{1}{2} & \alpha_j = \alpha_k \text{ or } \beta_j = \beta_k \text{ (but not both)}; \\ -\frac{1}{2} & \alpha_j = \beta_k \text{ or } \beta_j = \alpha_k \text{ (but not both)}; \\ 1 & \alpha_j = \alpha_k \text{ and } \beta_j = \beta_k; \\ -1 & \alpha_j = \beta_k \text{ and } \beta_j = \alpha_k. \end{cases}$$

About that weird covariance structure...

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Another Gaussian process that has it: Again suppose that $I_j := (e^{i\alpha_j}, e^{i\beta_j})$ are intervals on $\mathbb{S}^1$, and suppose that $\{G_\theta\}_{\theta\in[0,2\pi)}$ are i.i.d. standard Gaussians. Define

$$X_{n,k} = G_{\beta_k} - G_{\alpha_k};$$

then

$$\mathrm{Cov}(X_j, X_k) = \begin{cases} 0, & \alpha_j, \alpha_k, \beta_j, \beta_k \text{ all distict}; \\ \frac{1}{2} & \alpha_j = \alpha_k \text{ or } \beta_j = \beta_k \text{ (but not both)}; \\ -\frac{1}{2} & \alpha_j = \beta_k \text{ or } \beta_j = \alpha_k \text{ (but not both)}; \\ 1 & \alpha_j = \alpha_k \text{ and } \beta_j = \beta_k; \\ -1 & \alpha_j = \beta_k \text{ and } \beta_j = \alpha_k. \end{cases}$$

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Theorem (Hughes–Keating–O'Connel)

Let $Z(\theta)$ be the characteristic polynomial of U and fix $\theta_1 \ldots, \theta_k$. Then

$$\frac{1}{\sqrt{\frac{1}{2}\log(n)}}\big(\log(Z(\theta_1)), \ldots, \log(Z(\theta_k))\big)$$

converges in distribution to a standard Gaussian random vector in $\mathbb{C}^k$, as $n \to \infty$.

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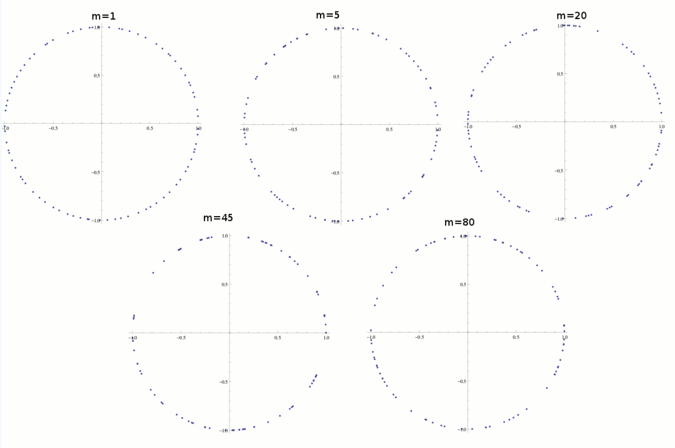
HKO in particular showed that Wieand's result follows from theirs by the argument principle.

Powers of U



The eigenvalues of U^m for $m = 1, 5, 20, 45, 80$, for U a realization of a random 80×80 unitary matrix.

Rains’ Theorems

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Theorem (Rains 1997)

Let $U \in \mathbb{U}(n)$ be a random unitary matrix, and let $m \geq n$. Then the eigenvalues of U^m are distributed exactly *as n i.i.d. uniform points on $\mathbb{S}^1$.*

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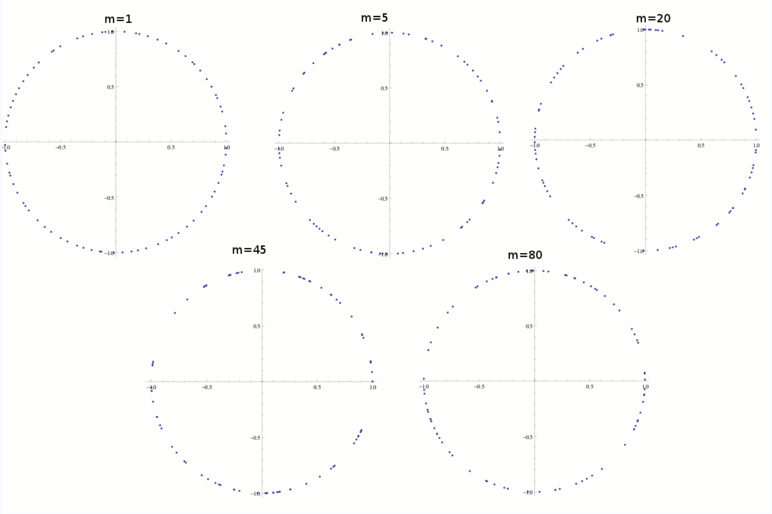
Theorem (Rains 2003)

Let $m \leq N$ be fixed. Then

$$[\mathbb{U}(N)]^m \overset{e.v.d.}{=} \bigoplus_{0 \leq j < m} \mathbb{U}\left(\left\lceil \frac{N-j}{m} \right\rceil \right),$$

where $\overset{e.v.d.}{=}$ denotes equality of eigenvalue distributions.



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Theorem (E.M./M. Meckes)

Let ν denote the uniform probability measure on the circle and

$$W_p(\mu,\nu) := \inf\left\{ \left(\int |x-y|^p \, d\pi(x,y)\right)^{\frac{1}{p}} \;\middle|\; \begin{array}{l} \pi(A\times\mathbb{C}) = \mu(A) \\ \pi(\mathbb{C}\times A) = \nu(A) \end{array} \right\}.$$

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Then

- $\mathbb{E}\left[W_p(\mu_{m,N},\nu)\right] \le \frac{Cp\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}}{N}$.

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Then

- $\mathbb{E}\left[W_p(\mu_{m,N},\nu)\right] \leq \frac{Cp\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}}{N}.$
- *For* $1 \leq p \leq 2$,
$$\mathbb{P}\left[W_p(\mu_{m,N},\nu) \geq \frac{C\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}}{N} + t\right] \leq \exp\left[-\frac{N^2t^2}{24m}\right].$$
- *For* $p > 2$,
$$\mathbb{P}\left[W_p(\mu_{m,N},\nu) \geq \frac{Cp\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}}{N} + t\right] \leq \exp\left[-\frac{N^{1+\frac{2}{p}}t^2}{24m}\right].$$

Almost sure convergence

Corollary

For each N, let U_N *be distributed according to uniform measure on* $\mathbb{U}(N)$ *and let* $m_N \in \{1, \ldots, N\}$*. There is a C such that, with probability 1,*

$$W_p(\mu_{m_N,N}, \nu) \leq \frac{Cp\sqrt{m_N \log(N)}}{N^{\frac{1}{2}+\frac{1}{\max(2,p)}}}$$

eventually.

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Theorem (Hough/Krishnapur/Peres/Virág 2006)

Let $\mathcal{X}$ be a determinantal point process in Λ satisfying some niceness conditions. For $D \subseteq \Lambda$, let $\mathcal{N}_D$ be the number of points of $\mathcal{X}$ in D. Then

$$\mathcal{N}_D \stackrel{d}{=} \sum_k \xi_k,$$

where $\{\xi_k\}$ are independent Bernoulli random variables with means given explicitly in terms of the kernel of $\mathcal{X}$.

A miraculous representation of the eigenvalue counting function

That is, if $\mathcal{N}_\theta$ is the number of eigenangles of U between 0 and θ, then

$$\mathcal{N}_\theta \stackrel{d}{=} \sum_{j=1}^{N} \xi_j$$

for a collection $\{\xi_j\}_{j=1}^{N}$ of independent Bernoulli random variables.

A miraculous representation of the eigenvalue counting function

Recall Rains' second theorem:

$$[\mathbb{U}(N)]^m \overset{e.v.d.}{=} \bigoplus_{0 \le j < m} \mathbb{U}\left(\left\lceil \frac{N-j}{m} \right\rceil\right),$$

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So: if $\mathcal{N}_{m,N}(\theta)$ denotes the number of eigenangles of U^m in $[0,\theta)$, then

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- $\operatorname{Var}\left[\mathcal{N}_{1,N}(\theta)\right] \leq \log(N) + 1$ (e.g., via explicit computation with the kernel of the determinantal point process), and so

$$\operatorname{Var}\left(\mathcal{N}_{m,N}(\theta)\right) = \sum_{0 \leq j < m} \operatorname{Var}\left(\mathcal{N}_{1,\left\lceil \frac{N-j}{m} \right\rceil}(\theta)\right) \leq m\left(\log\left(\frac{N}{m}\right) + 1\right).$$

The concentration of $\mathcal{N}_{m,N}$ leads to concentration of individual eigenvalues about their predicted values:

$$\mathbb{P}\left[\left|\theta_j - \frac{2\pi j}{N}\right| > \frac{4\pi t}{N}\right] \leq 4\exp\left[-\min\left\{\frac{t^2}{m\left(\log\left(\frac{N}{m}\right)+1\right)}, t\right\}\right],$$

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Bounding $\mathbb{E} W_p(\mu_{m,N}, \nu)$

If $\nu_N := \frac{1}{N} \sum_{j=1}^{N} \delta_{\exp\left(i \frac{2\pi j}{N}\right)}$, then $W_p(\nu_N, \nu) \leq \frac{\pi}{N}$ and

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$$\leq 8\Gamma(p+1) \left(\frac{4\pi \sqrt{m \left[\log\left(\frac{N}{m}\right) + 1 \right]}}{N} \right)^p,$$

using the concentration result and Fubini's theorem.

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- $F_p(U_1, \ldots, U_m)$ is Lipschitz (w.r.t. the L_2 sum of the Euclidean metrics) with Lipschitz constant $N^{-\frac{1}{\max(p,2)}}$.
- If we had a general concentration phenomenon on $\bigoplus_{0\le j<m} \mathbb{U}\left(\left\lceil \frac{N-j}{m} \right\rceil\right)$, concentration of $W_p\left(\mu_{U^m}, \nu\right)$ would follow.

Concentration on $\mathbb{U}(N_1) \oplus \cdots \oplus \mathbb{U}(N_k)$

Theorem (E. M./M. Meckes)

Given $N_1, \ldots, N_k \in \mathbb{N}$*, denote by* $M = \mathbb{U}(N_1) \times \cdots \mathbb{U}(N_k)$ *equipped with the* L_2*-sum of Hilbert–Schmidt metrics.*

Suppose that $F : M \to \mathbb{R}$ *is L-Lipschitz, and that* $U_j \in \mathbb{U}(N_j)$ *are independent, uniform random unitary matrices, for* $1 \leq j \leq k$*. Then for each* $t > 0$*,*

$$\mathbb{P}\Big[F(U_1, \ldots, U_k) \geq \mathbb{E}F(U_1, \ldots, U_k) + t\Big] \leq e^{-Nt^2/12L^2},$$

where $N = \min\{N_1, \ldots, N_k\}$*.*