Name: Solutions

Math 224 Exam 6
December 5, 2012

1. For $x, y > 0$, consider the system

$$\frac{dx}{dt} = \frac{1}{y}$$
$$\frac{dy}{dt} = \frac{a}{x},$$

where a is a real parameter.

15

(a) Show that the system is Hamiltonian and identify the Hamiltonian function.

Let $f(x,y) = \frac{1}{y}$, $g(x,y) = \frac{a}{x}$. Then $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} = 0$,

So the system is Hamiltonian. Moreover,

$$H(x,y) = \int \frac{1}{y}\, dy = \log(y) + \phi(x)$$

$$\Rightarrow -\frac{\partial H}{\partial x}(x,y) = -\phi'(x), \text{ and we need } -\frac{\partial H}{\partial x}(x,y) = \frac{a}{x}$$

$$\Rightarrow \phi(x) = -a \log x + c$$

$\Rightarrow$ the Hamiltonian of the system can be taken to be $H(x,y) = \log y - a \log x$

$$= \log\left[\frac{y}{x^a}\right].$$

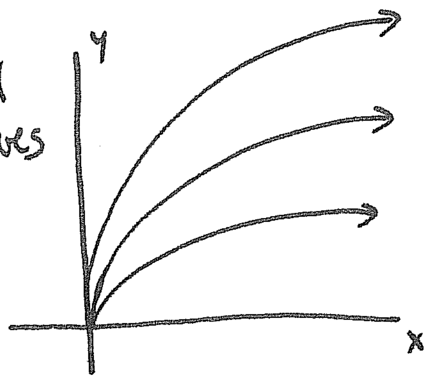
20 (b) Sketch the level curves of the system for $a = \frac{1}{2}, 1, 2$, and use them to draw the phase portraits of the corresponding systems. (Note: it will be useful to recall that $\ln(b) - \ln(c) = \ln\left(\frac{b}{c}\right)$ and that $b\ln(c) = \ln(c^b)$.)

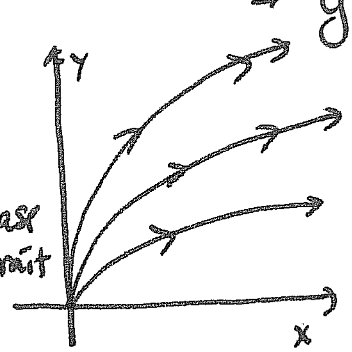
$a = 1/2$: $H(x,y) = \log\left[\frac{y}{\sqrt{x}}\right]$. If $H(x,y) = C$, then

$\frac{y}{\sqrt{x}} = e^{c}$ (also constant)

$\Rightarrow y = (e^{c})\sqrt{x}$

level curves



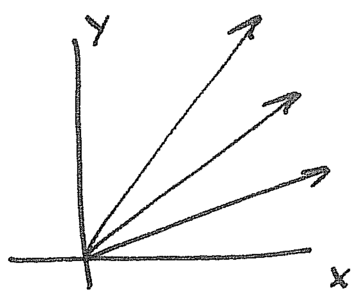


phase portrait

To get the phase portrait, we just need directions on the curves.

At $(1,1)$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt}$:

$a = 1$: $H(x,y) = \log\left[\frac{y}{x}\right]$. If $H(x,y) = C$, then $y = e^{c} x$

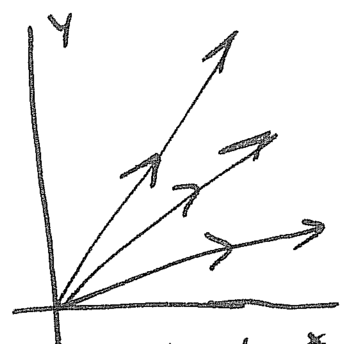


level curves

At $(1,1)$,

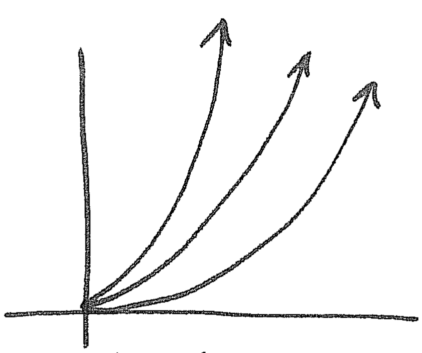
$\frac{dx}{dt} = 1$

$\frac{dy}{dt} = 1$



phase portrait

$a = 2$: $H(x,y) = \log\left[\frac{y}{x^2}\right]$. If $H(x,y) = C$, $y = (e^{c})\, x^2$

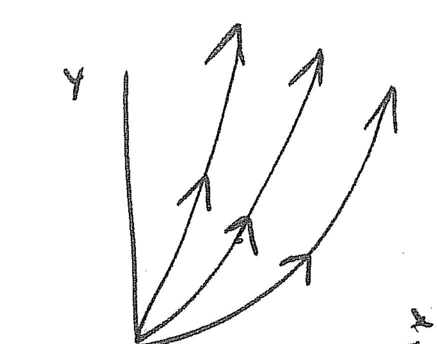


level curves

at $(1,1)$

$\frac{dx}{dt} = 1$

$\frac{dy}{dt} = 2$



phase portrait

2. Consider the forced harmonic oscillator

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 17y = 0; \qquad y(0) = 1, \quad y'(0) = 0.$$

(a) Is the system overdamped or underdamped?

15 Characteristic polynomial: $s^2 + 2s + 17$

$\Rightarrow$ eigenvalues : $\dfrac{-2 \pm \sqrt{4 - 4(17)}}{2} = -1 \pm 4i$

The system is underdamped.

(b) Suppose you hit the oscillator with a hammer at time $t = 3$. Modify the equation above to reflect this and then solve it.

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$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 17y = k\,\delta_3(t) \qquad \text{for some } k$$

$$\Rightarrow \quad s^2\mathcal{L}[y](s) - s + 2\mathcal{L}[y](s) - 2 + 17\mathcal{L}[y](s) = ke^{-3s}$$

$$\Rightarrow \quad \mathcal{L}[y](s) = \frac{s+2}{s^2+2s+17} + \frac{ke^{-3s}}{s^2+2s+17}$$

$$= \frac{s+1}{(s+1)^2 + 4^2} + \frac{1}{4}\left(\frac{4}{(s+1)^2+4^2}\right) + \frac{k}{4}\left[\frac{4e^{-3s}}{(s+1)^2+4^2}\right]$$

$$\Rightarrow y(t) = e^{-t}\cos(4t) + \frac{1}{4}e^{-t}\sin(4t) + \frac{k}{4}u_3(t)e^{-(t-3)}\sin(4(t-3))$$

5 (c) How would your answer change if you hit the oscillator twice as hard?

The final term would be multiplied by 2 (k is replaced by 2k).

3. Find the general solution to

30

$$\frac{dy}{dt} + 9y = u_2(t).$$

$$s\mathcal{L}[y] - y(0) + 9\mathcal{L}[y] = \frac{e^{-2s}}{s}$$

$$\Rightarrow \quad \mathcal{L}[y] = \frac{y(0)}{s+9} + \frac{e^{-2s}}{s(s+9)}$$

$$\frac{1}{s(s+9)} = \frac{A}{s} + \frac{B}{s+9} \quad \Rightarrow \quad A = \frac{1}{9}, \quad B = -\frac{1}{9}$$

$$\Rightarrow \quad \mathcal{L}[y](s) = \frac{y(0)}{s+9} + \frac{e^{-2s}}{9s} - \frac{e^{-2s}}{9(s+9)}$$

$$\Rightarrow \quad y(t) = y(0)\, e^{-9t} + \frac{1}{9} u_2(t) - \frac{1}{9} u_2(t) e^{-9(t-2)}$$

$$\mathcal{L}[y] = \int_0^\infty y(t)e^{-st}\,dt$$

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$$

$y(t)$	$Y(s) = \mathcal{L}[y]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u_a(t)$	$\frac{e^{-sa}}{s}$
δ_a	e^{-as}
$u_a(t)f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-\frac{dY}{ds}$