Name Solutions

Math 224 Exam 1
January 30, 2015

1. (a) Find the general solution of $\frac{dy}{dt} = 2ty$.

This is a separable equation, so we separate variables:

$$\int \frac{1}{y}\,dy = \int 2t\,dt$$

$$\ln|y| = t^2 + C$$

$$|y| = e^{t^2+C} = e^C e^{t^2}$$

$$y = \pm e^C e^{t^2}$$

Since $y=0$ is also a solution,

$y = k e^{t^2}$ is the general solution.

(b) Verify that $y(t) = (\sin t)e^{t^2}$ is a solution of

$$\frac{dy}{dt} = 2ty + (\cos t)e^{t^2}.$$

Just use the Rat Poison Principle:

$$\frac{d}{dt}\left[(\sin t)e^{t^2}\right] = (\cos t)\, e^{t^2} + (\sin t)\, e^{t^2} \cdot 2t$$

and $2ty + (\cos t)e^{t^2} = 2t(\sin t)e^{t^2} + (\cos t)e^{t^2}$,

so $\frac{dy}{dt} = 2ty + (\cos t)e^{t^2}$.

(c) Find the solution of the initial value problem

$$\frac{dy}{dt} = 2ty + (\cos t)e^{t^2}, \qquad y(0) = 4.$$

This is a linear equation.

In part (a) we found the general solution of the homogeneous equation $\frac{dy}{dt} = 2ty$: $y_h = ke^{t^2}$.

In (b) we saw that $y_p = (\sin t)\, e^{t^2}$ is a particular solution of the original equation.

So the general solution is $y = (\sin t)e^{t^2} + ke^{t^2}$.

By the initial condition, $4 = 0 + k \cdot 1$, so the solution to the IVP is $y = (\sin t + 4)\, e^{t^2}$.

2. Consider the modified logistic model

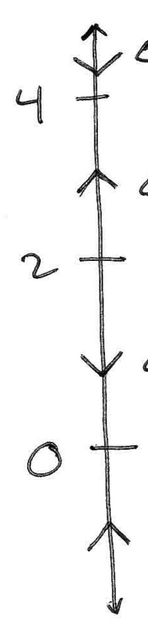
$$\frac{dP}{dt} = P\left(1 - \frac{P}{4}\right)\left(\frac{P}{2} - 1\right).$$

(a) Find all the equilibrium solutions of this differential equation.

The RHS is 0 when $P = 0, 4, 2$, so these are the equilibria.

(b) Draw the phase line for this equation, and identify the equilibria as sinks, sources, or nodes.

$5(1-\frac{5}{4})(\frac{5}{2}-1) < 0$



$3(1-\frac{3}{4})(\frac{3}{2}-1) > 0$

$1\cdot(1-\frac{1}{4})(\frac{1}{2}-1) < 0$

$-1(1+\frac{1}{4})(-\frac{1}{2}-1) > 0$

0 and 4 are sinks.

2 is a source

(c) What will be the long-term behavior of the population in each of the following situations?

i. $P(0) = 1$

The population will decrease and approach 0.

ii. $P(0) = 3$

The population will increase and approach 4.

iii. $P(0) = 5$

The population will decrease and approach 4.

(d) What is the significance, for a population described by this model, of the population size 2?

A population larger than 2 (even only a little larger) will survive, whereas a population smaller than 2 (even only a little smaller) will not. So 2 is a threshold value, where the long-term fate of the population changes drastically.

3. (a) Verify that $y_1(t) = \frac{1}{\sqrt{2t+1}}$ and $y_2(t) = \frac{1}{\sqrt{2t+4}}$ are both solutions of

$$\frac{dy}{dt} = -y^3.$$

Use the Rat Poison Principle:

$$\frac{dy_1}{dt} = \frac{d}{dt}(2t+1)^{-1/2} = -\frac{1}{2}(2t+1)^{-3/2} \cdot 2$$

$$= -\frac{1}{(2t+1)^{3/2}} = -y_1^3$$

and $$\frac{dy_2}{dt} = \frac{d}{dt}(2t+4)^{-1/2} = -\frac{1}{2}(2t+4)^{-3/2} \cdot 2$$

$$= -\frac{1}{(2t+4)^{3/2}} = -y_2^3.$$

(b) *Without* finding the general solution of the differential equation, what can you say about solutions of $\frac{dy}{dt} = -y^3$ for which the initial condition $y(0)$ satisfies $\frac{1}{2} < y(0) < 1$?

$y_1(0) = 1$ and $y_2(0) = \frac{1}{2}$, so

$y_2(0) < y(0) < y_1(0)$.

By the Uniqueness Theorem, solution curves don't cross, so $y_2(t) < y(t) < y_1(t)$ for all t.

Either from this or the phase line, we can see that $y \longrightarrow 0$ as $t \to \infty$.

Moreover, since $y_2(t) \to \infty$ as $t \to -2^+$, we can tell that y becomes undefined for some $t \geq -2$.

4. (a) Use Euler's method with $\Delta t = 1$ to approximate the solution of the initial value problem $\frac{dy}{dt} = (2-y)(y+1)$, $y(0) = 1$ at $t = 3$.

$t_{k+1} = t_k + \Delta t, \quad y_{k+1} = y_k + \Delta t\,(2-y_k)(y_k+1)$

$t_0 = 0 \quad y_0 = 1$

$t_1 = 1 \quad y_1 = 1 + 1\cdot(2-1)(1+1) = 3$

$t_2 = 2 \quad y_2 = 3 + 1\cdot(2-3)(1+3) = -1$

$t_3 = 3 \quad y_3 = -1 + 1\cdot(2+1)(-1+1) = -1$

(b) Does your approximation seem reasonable? Why or why not?

No, because it jumps back and forth across the equilibrium at 2 and settles at a different equilibrium.

In fact we know the correct solution approaches 2 (from the phase line).