Name: Solutions

Math 224 Practice Quiz 2

1. Solve the system

$$\frac{dx}{dt} = 3x + y,$$
$$\frac{dy}{dt} = -y$$

with initial conditions $x(0) = 1$, $y(0) = 2$.

Start with $\frac{dy}{dt} = -y \Rightarrow y(t) = k_y e^{-t}$.

Now we get an inhomogeneous equation for x:

$$\frac{dx}{dt} = 3x + k_y e^{-t}.$$

Solving the homogeneous part gives $x(t) = k_x e^{3t}$.

Now use variation of parameters: $x'(t) = k_x'(t) e^{3t} + 3k_x(t) e^{3t}$

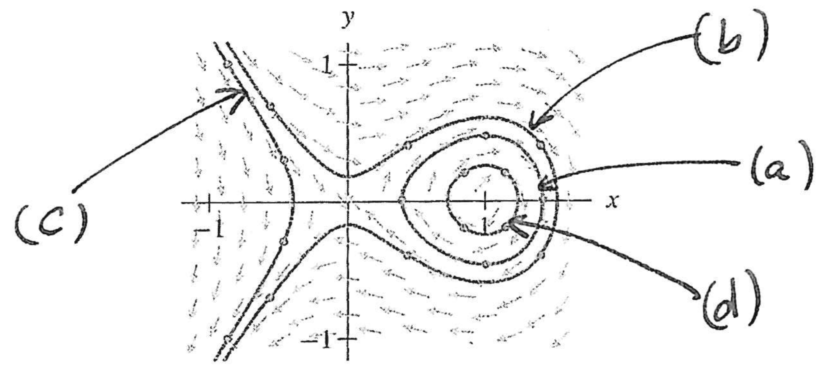
$$= k_x'(t) e^{3t} + 3x(t).$$

So we need $k_x'(t) \cdot e^{3t} = k_y e^{-t} \Leftrightarrow k_x'(t) = k_y e^{-4t}$

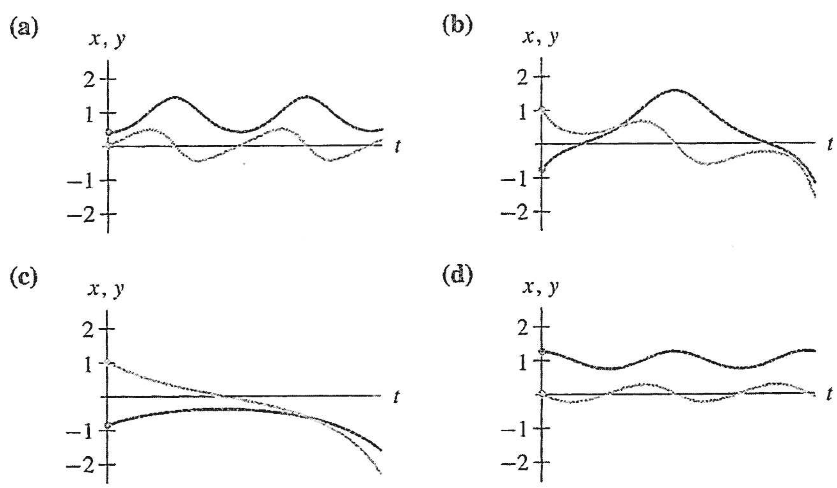
$$\Rightarrow k_x(t) = \frac{k_y e^{-4t}}{(-4)} + C$$

$\Rightarrow x(t) = -\frac{1}{4} k_y e^{-t} + Ce^{3t}$ and $y(t) = k_y e^{-t}$ is the general solution. We want $x(0) = -\frac{k_y}{4} + C = 1$ and $y(0) = k_y = 2 \Rightarrow C = \frac{3}{2}$. So the solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{3t} \\ 2e^{-t} \end{pmatrix}$

2. The following is a graph of four solution curves $(x(t), y(t))$ to an autonomous system of differential equations, together with the direction field of the system.



Below are four pairs of graphs of $x(t)$ and $y(t)$ versus t.



Match each of the pairs of graphs to a solution curve in the phase plane. Label which graph is x and which is y. Finally, describe the long-term behavior of solutions in all cases.

On curve (a), x oscillates around 1 and y oscillates around 0. This is also what happens with curve (d), just with smaller oscillations. For (b), x increases to a positive max., then tends to $-\infty$, while y initially decreases, then goes up a little, down, back up, then down to $-\infty$. On curve (c), x_2 stays negative & decreases to $-\infty$; y tends to $-\infty$ as well.

3. Consider the linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{B}\mathbf{Y} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \mathbf{Y}$.

(a) Find the eigenvalues of B.

The characteristic polynomial is $\lambda^2 - tr(B)\lambda + \det B$
$= \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1)$

So the eigenvalues are
$\lambda_1 = -3$, $\lambda_2 = -1$.

(b) Find the corresponding eigenvectors.

$\lambda_1 = -3$: We need a solution to $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Take $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda_2 = -1$: We need a solution to $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Take $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(c) Give the general solution to the system.

$$Y(t) = k_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(d) Solve the initial value problem $\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \mathbf{Y}$ and $\mathbf{Y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{array}{l} k_1 + k_2 = 1 \\ k_1 - k_2 = 0 \end{array}$$

$$\Rightarrow k_1 = k_2 = 1.$$

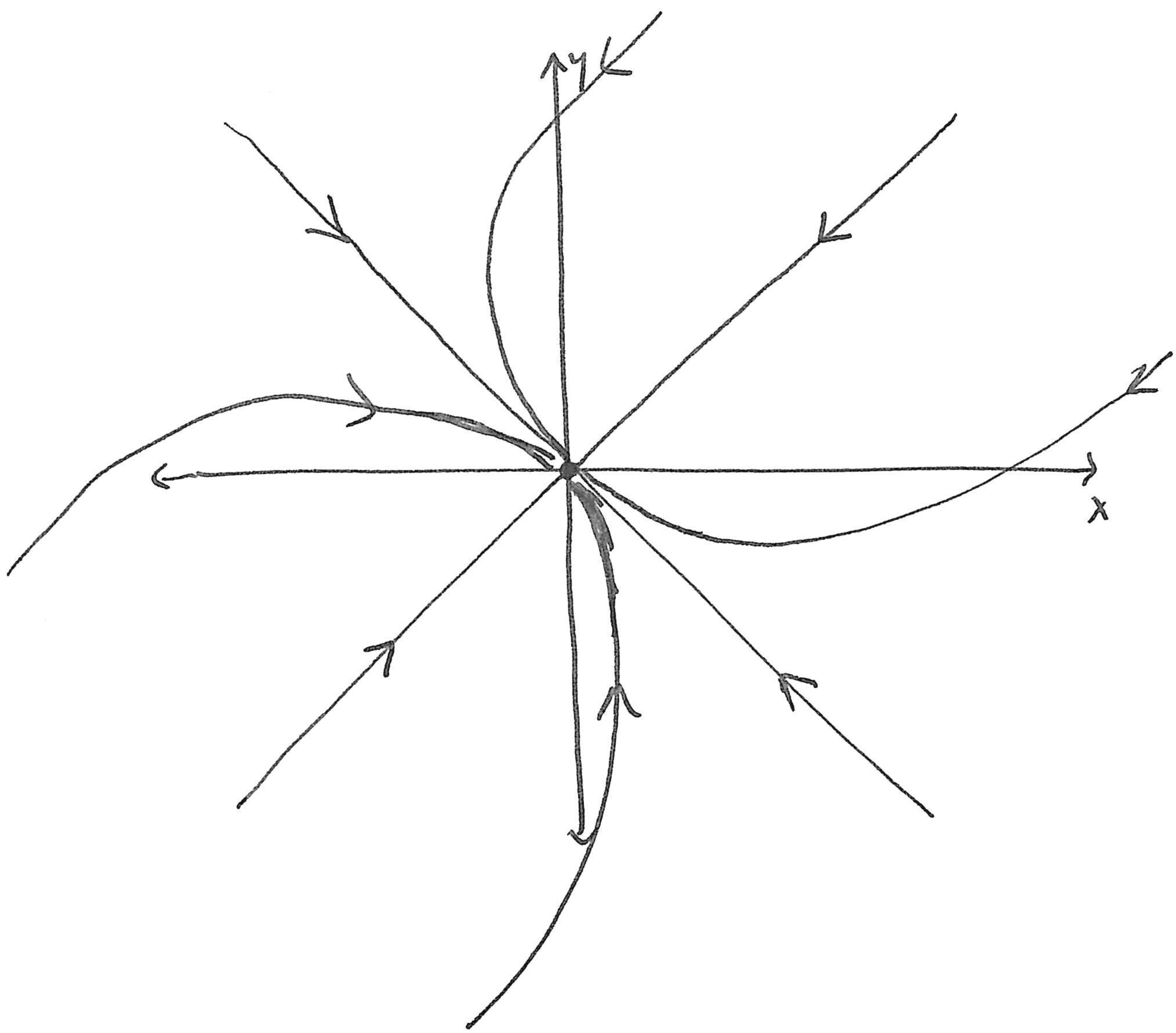
$$Y(t) = e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(e) What is the long-term behavior of your solution as $t \to \infty$? What about $t \to -\infty$?

As $t \to \infty$, both $x(t)$ and $y(t) \to 0$.
The curve approaches 0 tangent to the line $y = -x$.

As $t \to \infty$, both $x(t)$ and $y(t) \to +\infty$.
The curve is parallel to the line $y = x$.

(f) Sketch the phase plane for this system. Make sure to include any straight-line solutions, indicate direction of solution curves in time, and include the solution curve you found above to the initial value problem.



4. Suppose you used Euler's method to approximate the solution to the **autonomous** system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

with initial condition $\mathbf{Y}(0) = \mathbf{Y}_0$, and the resulting solution curve plotted on the phase plane looked like this:

(a) Explain how you can tell that the Euler's method approximation must not be a very good approximation of the true solution.

The system is autonomous, but the approximate solution crosses itself, which is not possible for the true solution.

(b) What would you do to try to get a better approximation?

Smaller Δt is probably a good idea.

(c) What do you guess the true solution looks like, based on the approximation above?

Maybe it's periodic? Or spirals in ~~or out~~ toward something periodic?
Or spirals out toward something periodic?