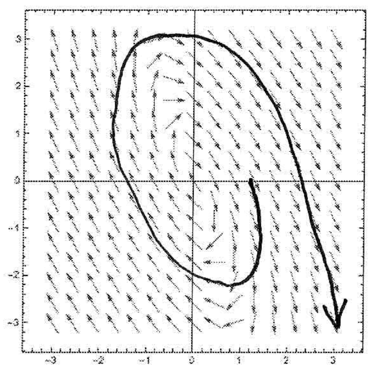
Name: Solutions

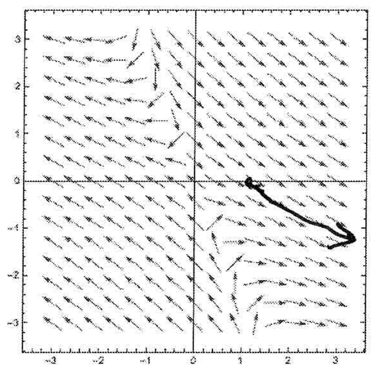
Math 224 Exam 3
March 6, 2015

1. Here are two different versions of the model for Paul and Bob's cafés, both of which suppose that current profits from either café have a positive effect on Paul's profits, and current profits from either café have a negative effect on Bob's profits:

$$\frac{dx}{dt} = 5x + 2y \qquad\qquad \frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = -3x - 2y \qquad\qquad \frac{dy}{dt} = -4x - y.$$

(a) Identify which of the following two direction fields is for which system. Justify your answer.





Check the direction fields at $(1,0)$:

$\begin{pmatrix} 5 & 2 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ ← shallower

$\begin{pmatrix} 2 & 1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ ← steeper

Based on this test, the first picture is for the 2nd system and vice versa

or: the char. polys are

$\det\begin{bmatrix} 5-\lambda & 2 \\ -3 & -2-\lambda \end{bmatrix} = (-2-\lambda)(5-\lambda) + 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$

(real ews; saddle - 2nd pict)

$\det\begin{bmatrix} 2-\lambda & 1 \\ -4 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda) + 4 = \lambda^2 - \lambda + 2$

$\lambda = \frac{1 \pm \sqrt{-7}}{2}$ (complex ews): spiral (first pict)

(b) For each system, sketch the solution curve with initial condition $x(0) = 1$, $y(0) = 0$ on the direction field above. Describe the long-term behavior of the solutions.

The first system's solution follows the curve in the second picture: $x \to \infty$, $y \to -\infty$, and the curve in the phase plane is asymptotic to a straight line.

The second system's solution spirals outward: x and y both oscillate with increasing amplitude.

2. Consider the system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$, where $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$.

5 (a) Find the eigenvalues of $\mathbf{A}$.

$$\det \begin{bmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) + 2$$

$$= \lambda^2 - 4\lambda + 5$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \boxed{2 \pm i}$$

5 (b) Determine from the eigenvalues alone what type of equilibrium the system has at the origin.

Complex eigenvalues with positive real part $\Rightarrow$ $\boxed{\text{Spiral source}}$

(c) Find the general solution of the system.

Find an eigenvector for $2+i$:

$$\begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad : \quad \begin{bmatrix} 1-i \\ -1 \end{bmatrix} \text{ works}$$

(easy to see for ~~top~~ bottom equation. For the top:

$(-1-i)(1-i) + (-2)(-1) = -2+i-i+2 = 0$)

So $Y_c(t) = e^{2t} e^{it} \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$ is a complex solution

$$Y_c(t) = e^{2t}(\cos t + i\sin t)\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$$

$$= e^{2t}\left(\cos t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) + ie^{2t}\left(\cos t\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

$\Rightarrow$ $Y_1(t) = e^{2t}\begin{pmatrix} \cos t + \sin t \\ -\cos t \end{pmatrix}$ and $Y_2(t) = e^{2t}\begin{pmatrix} -\cos t + \sin t \\ -\sin t \end{pmatrix}$

are solutions (and are linearly independent at $t=0$)

$$\Rightarrow \quad Y_{gen'l}(t) = k_1 e^{2t}\begin{pmatrix} \cos t + \sin t \\ -\cos t \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} -\cos t + \sin t \\ -\sin t \end{pmatrix}.$$

(d) Find the solution of the system with the initial condition $\mathbf{Y}(0) = (1, 0)$.

We need to find k_1, k_2 s.t.

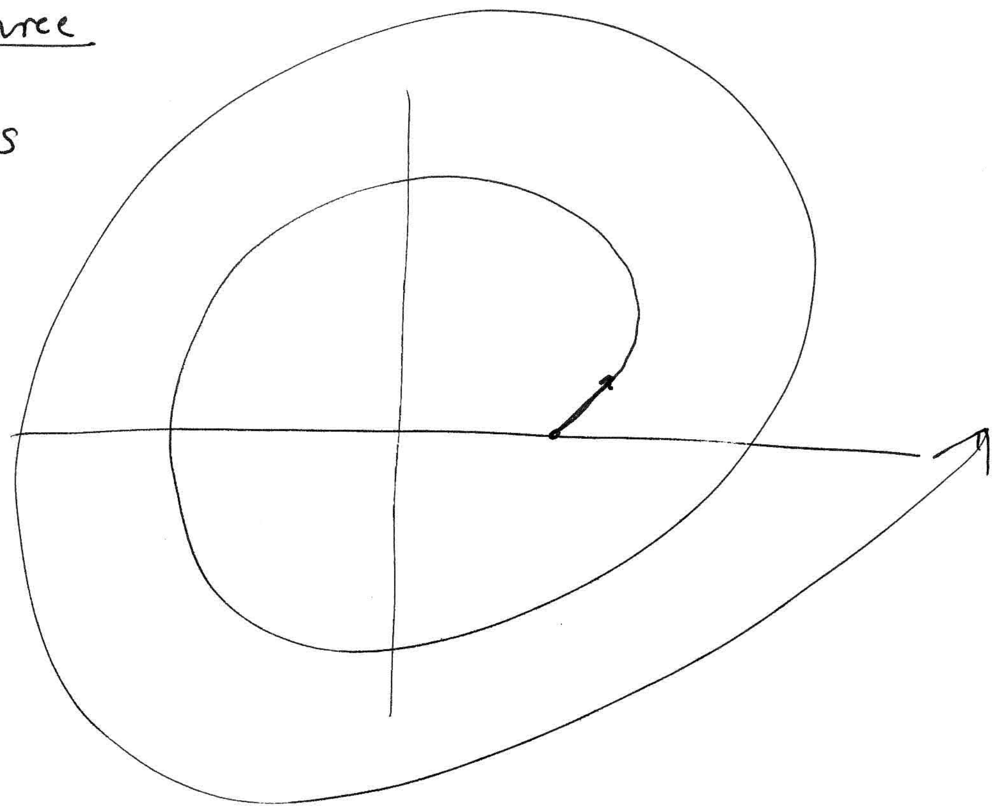
$k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Clearly $k_1 = 0$, $k_2 = -1$ works:

$$\boxed{Y(t) = e^{2t}\begin{pmatrix} -\sin t + \cos t \\ \sin t \end{pmatrix}}$$

(e) Sketch the phase portrait, including the solution curve with the initial condition $\mathbf{Y}(0) = (1, 0)$.

$$\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The ~~eq~~ equilibrium at 0 is a spiral source since the eigenvalues are $2 \pm i$.



3. Consider the system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$, where $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$.

(a) Find the eigenvalues of $\mathbf{A}$.

5

$$\det \begin{bmatrix} 3-\lambda & 4 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$$

$\Rightarrow$ eigenvalues : 4, -1

(b) Determine from the eigenvalues alone what type of equilibrium the system has at the origin.

5

One positive, one negative eigenvalue

$\Rightarrow$ $\boxed{\text{Saddle}}$

(c) Find the general solution of the system.

15

$\lambda = 4$: $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ take $V_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\lambda = -1$ $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ take $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\Rightarrow \boxed{Y_{gen}(t) = k_1 e^{4t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

(d) Find the solution of the system with the initial condition $\mathbf{Y}(0) = (1, 0)$.

5

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = k_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Leftrightarrow$ $\begin{matrix} 4k_1 + k_2 = 1 \\ k_1 - k_2 = 0 \end{matrix}$

$\Rightarrow$ $5k_1 = 1 \Rightarrow k_1 = \frac{1}{5} \Rightarrow k_2 = \frac{1}{5}$

$$\Rightarrow \boxed{Y(t) = \frac{1}{5} e^{4t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \frac{1}{5} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

(e) Sketch the phase portrait, including the solution curve with the initial condition $\mathbf{Y}(0) = (1, 0)$.

10