Name: ______________________________

Math 224 Exam 6

April 29, 2013

1. Consider the forced harmonic oscillator

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 17y = 0; \qquad y(0) = 1, \quad y'(0) = 0.$$

   (a) Is the system overdamped or underdamped?

   (b) Suppose you hit the oscillating block with a hammer (on the spring side) at time $t = 3$, with total force 4. Modify the equation above to reflect this and then solve it.

(c) How would your answer change if you hit it with the same force at the same time, but in the opposite direction?

(d) Show that either way, the velocity of the block is discontinuous at time $t = 3$.

2. Consider the initial value problem

$$\frac{dy}{dt} = f(t, y) = 3t, \qquad y(0) = 2.$$

(a) Using $\Delta t = \frac{1}{2}$, fill out the table below and give the resulting improved Euler's method estimate for $y(1)$:

k	t_k	y_k	$m_k := f(t_k, y_k)$	t_{k+1}	y^*_{k+1}	$n_{k+1} := f(t_{k+1}, y^*_{k+1})$	$\frac{m_k+n_{k+1}}{2}$	y_{k+1}
0	0	2						
1	$\frac{1}{2}$							

(b) Solve the initial value problem analytically and compute $y(1)$.

(c) What is the error in your approximation of $y(1)$?

(d) Graph your approximation of the solution and the solution itself on the same axes.

(e) Using our geometric derivation of improved Euler's method, explain why your approximate solution and your exact solution agree at $t = \frac{1}{2}$ and at $t = 1$ (but not in between).

Laplace transforms:

$$\mathcal{L}[y] = \int_0^\infty y(t)e^{-st}\ dt$$

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$$

$y(t)$	$Y(s) = \mathcal{L}[y]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u_a(t)$	$\frac{e^{-sa}}{s}$
δ_a	e^{-as}
$u_a(t)f(t-a)$	$e^{-as}\mathcal{L}[f](s)$
$e^{at}f(t)$	$\mathcal{L}[f](s-a)$
$tf(t)$	$-\frac{d}{ds}\left(\mathcal{L}[f]\right)$

Numerical Methods

Euler: $y_{k+1} = y_k + \Delta t f(t_k, y_k)$

Improved Euler:

- $m_k = f(t_k, y_k)$
- $y^*_{k+1} = y_k + (\Delta t)m_k$
- $n_{k+1} = f(t_{k+1}, y^*_{k+1})$
- $y_{k+1} = y_k + (\Delta t)\left(\frac{m_k + n_{k+1}}{2}\right)$