Name: Solutions

Math 224 Exam 6
April 29, 2013

1. Consider the forced harmonic oscillator

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 17y = 0; \qquad y(0) = 1, \quad y'(0) = 0.$$

(a) Is the system overdamped or underdamped?

The characteristic polynomial is $s^2+2s+17$, with roots $\frac{-2 \pm \sqrt{4-4\cdot 17}}{2}$

$= -1 \pm 4i$.

The system is therefore underdamped.

(b) Suppose you hit the oscillating block with a hammer (on the spring side) at time $t = 3$, with total force 4. Modify the equation above to reflect this and then solve it.

Hitting the block on the spring side means a force is applied in the positive direction:

$y'' + 2y' + 17y = 4\delta_3(t)$.

Laplace transform both sides:

$s^2 \mathcal{L}[y](s) - s + 2s\mathcal{L}[y](s) - 1 + 17\mathcal{L}[y](s) = 4e^{-3t}$

$\Rightarrow \mathcal{L}[y](s) = \frac{s+1}{s^2+2s+17} + 4e^{-3t}\frac{1}{s^2+2s+17}$

$= \frac{s+1}{(s+1)^2+16} + e^{-3t}\frac{4}{(s+1)^2+16}$

$= \mathcal{L}[\cos(4t)](s+1) + e^{-3t}\mathcal{L}[\sin(4t)](s+1)$

$$\boxed{y(t) = e^{-t}\cos(4t) + u_3(t)\, e^{-(t-3)} \sin(4(t-3))}$$

(c) How would your answer change if you hit it with the same force at the same time, but in the opposite direction?

This would only change the forcing function to $-4\delta_3(t)$: the solution becomes

$$y(t) = e^{-t}\cos(4t) - u_3(t)e^{-(t-3)}\sin(4(t-3)).$$

(d) Show that either way, the velocity of the block is discontinuous at time $t = 3$.

Starting from the formula

$$y(t) = e^{-t}\cos(4t) + u_3(t)e^{-(t-3)}\sin(4(t-3)),$$

for $t \neq 3$ we get

$$V(t) = y'(t) = -e^{-t}\cos(4t) - 4e^{-t}\sin(4t)$$

$$+ u_3(t)\left[-e^{-(t-3)}\sin(4(t-3)) + 4e^{-(t-3)}\cos(4(t-3))\right].$$

The last term is discontinuous at $t=3$:

$$4e^{-(3-3)}\cos(4(3-3)) = 4,$$ so

$4u_3(t)e^{-(t-3)}\cos(4(t-3))$ has a jump discontinuity at $t=3$. The same term appears in the velocity of part (c), just with a $-$ sign.

2. Consider the initial value problem

$$\frac{dy}{dt} = f(t, y) = 3t, \qquad y(0) = 2.$$

(a) Using $\Delta t = \frac{1}{2}$, fill out the table below and give the resulting improved Euler's method estimate for $y(1)$:

| $k$ | $t_k$ | $y_k$ | $m_k := f(t_k, y_k)$ | $t_{k+1}$ | $y^*_{k+1}$ | $n_{k+1} := f(t_{k+1}, y^*_{k+1})$ | $\frac{m_k+n_{k+1}}{2}$ | $y_{k+1}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | $0$ | $\frac{1}{2}$ | $2$ | $\frac{3}{2}$ | $\frac{3}{4}$ | $2\frac{3}{8}$ |
| 1 | $\frac{1}{2}$ | $\frac{19}{8}$ | $\frac{3}{2}$ | $1$ | $\frac{25}{8}$ | $3$ | $\frac{9}{4}$ | $\frac{28}{8} = \frac{7}{2}$ |

(b) Solve the initial value problem analytically and compute $y(1)$.

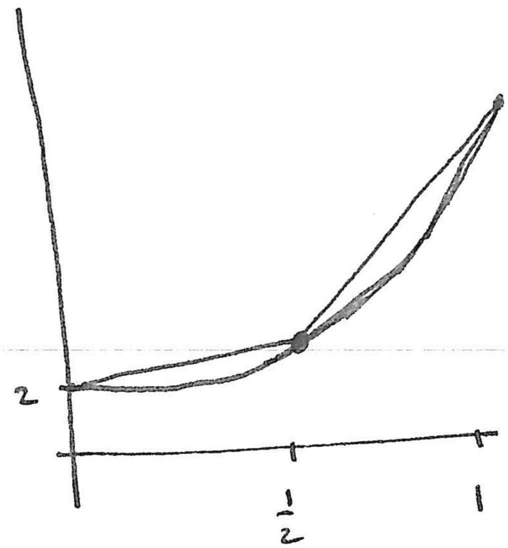
$\frac{dy}{dt} = 3t \Rightarrow y(t) = \frac{3}{2}t^2 + C \qquad y(0) = 2 = C$

$\Rightarrow y(t) = \frac{3}{2}t^2 + 2$, and $y(1) = \frac{7}{2}$

(c) What is the error in your approximation of $y(1)$?

$y(1) - y_2 = 0.$

(d) Graph your approximation of the solution and the solution itself on the same axes.




(e) Using our geometric derivation of improved Euler's method, explain why your approximate solution and your exact solution agree at $t = \frac{1}{2}$ and at $t = 1$ (but not in between).

Since the slopes $m_0$ and $n_0$ depend only on $t$, not on $y$, they are both the exact slopes of the curves, and so $\frac{m_0+n_0}{2}$ is exactly the average slope of the solution over $[0, \frac{1}{2}]$ (and similarly $\frac{m_1+n_1}{2}$ is exactly the average slope over $[\frac{1}{2}, 1]$). This means $y(t_1) = y_1$ and $y(t_2) = y_2$. But the approximate solution is piecewise linear, so it doesn't match otherwise.

Laplace transforms:

$$\mathcal{L}[y] = \int_0^\infty y(t)e^{-st}\,dt$$

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$$

| $y(t)$ | $Y(s) = \mathcal{L}[y]$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\sin\omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos\omega t$ | $\frac{s}{s^2+\omega^2}$ |
| $u_a(t)$ | $\frac{e^{-sa}}{s}$ |
| $\delta_a$ | $e^{-as}$ |
| $u_a(t)f(t-a)$ | $e^{-as}\mathcal{L}[f](s)$ |
| $e^{at}f(t)$ | $\mathcal{L}[f](s-a)$ |
| $tf(t)$ | $-\frac{d}{ds}\left(\mathcal{L}[f]\right)$ |

## Numerical Methods

Euler: $y_{k+1} = y_k + \Delta t f(t_k, y_k)$
Improved Euler:

- $m_k = f(t_k, y_k)$
- $y^*_{k+1} = y_k + (\Delta t)m_k$
- $n_{k+1} = f(t_{k+1}, y^*_{k+1})$
- $y_{k+1} = y_k + (\Delta t)\left(\frac{m_k+n_{k+1}}{2}\right)$